

A Rudimentary Quantum Compiler (2cnd Ed.)

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February 1, 2008

Abstract

We present a new algorithm for reducing an arbitrary unitary matrix U into a sequence of elementary operations (operations such as controlled-nots and qubit rotations). Such a sequence of operations can be used to manipulate an array of quantum bits (i.e., a quantum computer). Our algorithm applies recursively a mathematical technique called the CS Decomposition to build a binary tree of matrices whose product, in some order, equals the original matrix U . We show that the Fast Fourier Transform (FFT) algorithm is a special case of our algorithm. We report on a C++ program called “Qubiter” that implements the ideas of this paper. Qubiter(PATENT PENDING) source code is publicly available.

1. Introduction

1(a) Previous Work

In classical computation and digital electronics, one deals with sequences of elementary operations (operations such as AND, OR and NOT). These sequences are used to manipulate an array of classical bits. The operations are elementary in the sense that they act on only a few bits (1 or 2) at a time. Henceforth, we will sometimes refer to sequences as products and to operations as operators, matrices, instructions, steps or gates. Furthermore, we will abbreviate the phrase “sequence of elementary operations” by “SEO”. In quantum computation[1], one also deals with SEOs (with operations such as controlled-nots and qubit rotations), but for manipulating quantum bits (qubits) instead of classical bits. Quantum SEOs are often represented graphically by qubit circuits.

In quantum computation, one often knows the unitary operator U that describes the evolution of an array of qubits. One must then find a way to reduce U into a SEO. In this paper, we present a new algorithm for accomplishing this task. We also report on a C++ program called “Qubiter” that implements our algorithm. We call Qubiter a “quantum compiler” because, like a classical compiler, it produces a SEO for manipulating bits. Qubiter(PATENT PENDING) source code is publicly available at www.ar-tiste.com/qubiter.html.

Our algorithm can be applied to any unitary operator U .

It is useful to define certain unitary operators U_{N_B} for all $N_B \in \{1, 2, 3, \dots\}$, where U_{N_B} is a $2^{N_B} \times 2^{N_B}$ matrix and N_B is the number of bits. Some U_{N_B} are known to be expressible as a SEO whose length (i.e., whose number of elementary operations) is a polynomial in N_B . Two examples are the N_B bit Hadamard Transform (HT) matrix and the N_B bit Discrete Fourier Transform (DFT) matrix. The HT matrix is known to be expressible as a SEO of length $\text{Order}(N_B)$. The DFT matrix is known to be expressible (using the FFT algorithm of [2]-[4]) as a SEO of length $\text{Order}(N_B^2)$. Our algorithm achieves both of these SEO-length benchmarks. Even better, the SEO often called the “quantum FFT algorithm” is exactly reproduced by our algorithm.

We believe our algorithm yields short SEOs for many kinds of unitary operators other than the HT and DFT ones. We do not believe that our present algorithm yields the shortest possible SEO for every unitary operator. However, we do believe that it is possible to come very close to achieving this goal by introducing further optimizations into the algorithm. Future papers will report on our progress in finding such optimizations.

Previous workers [5] have described another algorithm for reducing a unitary operator into a SEO. Like ours, their algorithm can be applied to any unitary operator U . However, it is very unlikely that their algorithm will be efficient in producing short SEOs unless further optimizations are added to it. And such optimizations, if they exist, have not been specified by anyone. Furthermore, as far as we know, there

is no publicly available software that implements their algorithm. Our algorithm is significantly different from theirs. Theirs is based on a mathematical technique described in Refs.[6]-[7], whereas ours is based on a mathematical technique called the CS Decomposition (CSD)[8]-[12] to be described later.

Quantum Bayesian (QB) Nets[13]-[14] are a method of modeling quantum systems graphically in terms of network diagrams. In a companion paper[15], we show how to apply the results of this paper to QB nets.

1(b) CS Decomposition

As mentioned earlier, our algorithm utilizes a mathematical technique called the CS Decomposition (CSD)[8]-[12]. The C and S stand for “cosine” and “sine”, respectively. Next we will state the special case of the CSD Theorem that arises in our algorithm.

Suppose that U is an $N \times N$ unitary matrix, where N is an even number. Then the CSD Theorem states that one can always express U in the form

$$U = \begin{bmatrix} L_0 & 0 \\ 0 & L_1 \end{bmatrix} D \begin{bmatrix} R_0 & 0 \\ 0 & R_1 \end{bmatrix}, \quad (1b.1)$$

where the left and right side matrices L_0, L_1, R_0, R_1 are $\frac{N}{2} \times \frac{N}{2}$ unitary matrices and

$$D = \begin{bmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{bmatrix}, \quad (1b.2a)$$

$$D_{00} = D_{11} = \text{diag}(C_1, C_2, \dots, C_{\frac{N}{2}}), \quad (1b.2b)$$

$$D_{01} = \text{diag}(S_1, S_2, \dots, S_{\frac{N}{2}}), \quad (1b.2c)$$

$$D_{10} = -D_{01}. \quad (1b.2d)$$

For all $i \in \{1, 2, \dots, \frac{N}{2}\}$, $C_i = \cos \theta_i$ and $S_i = \sin \theta_i$ for some angle θ_i . Given any CSD of U , it is easy to find (see Appendix A) another CSD of U for which the angles θ_i are in non-decreasing order and they are contained in the interval $[0, 90^\circ]$. Henceforth, we will assume that the angles θ_i are so ordered and in this range. We will use the term *D matrix* to refer to any matrix that satisfies Eqs.(1b.2). If one partitions U into four blocks U_{ij} of size $\frac{N}{2} \times \frac{N}{2}$, then

$$U_{ij} = L_i D_{ij} R_j, \quad (1b.3)$$

for $i, j \in \{0, 1\}$. Thus, D_{ij} gives the singular values[16] of U_{ij} .

More general versions of the CSD Theorem allow for the possibility that we partition U into 4 blocks of unequal size.

Note that if U were a general (not necessarily unitary) matrix, then the four blocks U_{ij} would be unrelated. Then to find the singular values of the four blocks U_{ij} would require eight unitary matrices (two for each block), instead of the four L_i, R_j . This double use of the L_i, R_j is a key property of the CSD.

1(c) Bird's Eye View of Algorithm

Our algorithm is described in detail in subsequent sections. Here we will only give a bird's eye view of it.

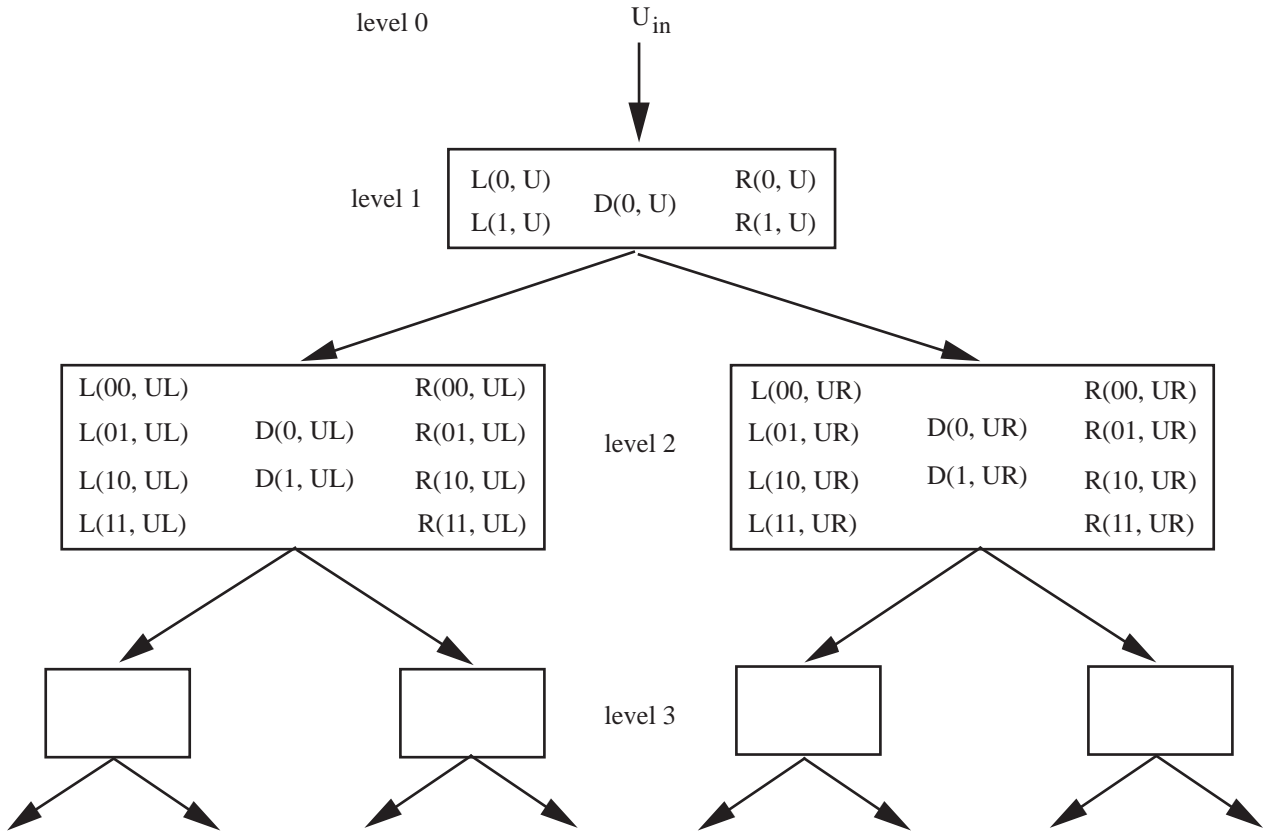


Fig.1 A CSD binary tree.

Consider Fig.1. We start with an initial unitary matrix U_{in} at horizontal level 0. Without loss of generality, we can assume that the dimension of U_{in} is 2^{N_B} for some $N_B \geq 1$. (If initially U_{in} 's dimension is not a power of 2, we replace it by a direct sum $U_{in} \oplus I_r$ whose dimension is a power of two.) We apply the CSD method to U_{in} . This yields for level 1 a D matrix $D(0, U)$, two unitary matrices $L(0, U)$ and $L(1, U)$ on the left side and two unitary matrices $R(0, U)$ and $R(1, U)$ on the right side. Then we apply the CSD method to each of the 4 matrices $L(0, U), L(1, U), R(0, U)$ and $R(1, U)$ that were produced in the previous step. Then we apply the CSD method

to each of the 16 R and L matrices that were produced in the previous step. And so on. Each node \mathcal{N} has children \mathcal{N}_L and \mathcal{N}_R to its left and right, respectively. \mathcal{N}_R (ditto, \mathcal{N}_L) has D, R, L matrices produced by applying the CSD method to the R matrices (ditto, L matrices) of its parent node \mathcal{N} . At level N_B , the L 's and R 's are 1×1 dimensional—i.e., just unit-modulus complex numbers. The nodes of the last level $N_B + 1$ don't have R, L or D matrices per se. If \mathcal{N}_R (ditto, \mathcal{N}_L) is a node of level $N_B + 1$, then it stores the R (ditto, L) matrices of its parent node \mathcal{N} .

Call a *central matrix* either (1) a single D matrix, or (2) a direct sum $D_1 \oplus D_2 \oplus \dots \oplus D_r$ of D matrices, or (3) a diagonal unitary matrix. From Fig.1 it is clear that the initial matrix U_{in} can be expressed as a product of central matrices, with each node of the tree providing one of the central matrices in the product. Later on we will present techniques for decomposing any central matrix into a SEO.

2. Preliminaries

In this section, we introduce some notation and some general mathematical concepts that will be used in subsequent sections.

2(a) General Notation

We define $Z_{a,b} = \{a, a + 1, \dots, b\}$ for any integers a and b . $\delta(x, y)$ equals one if $x = y$ and zero otherwise.

We will use the symbol N_B for the number (≥ 1) of bits and $N_S = 2^{N_B}$ for the number of states with N_B bits. Let $Bool = \{0, 1\}$. We will use lower case Latin letters $a, b, c \dots \in Bool$ to represent bit values and lower case Greek letters $\alpha, \beta, \gamma, \dots \in Z_{0, N_B-1}$ to represent bit positions. A vector such as $\vec{a} = a_{N_B-1} \dots a_2 a_1 a_0$ will represent a string of bit values, a_μ being the value of the μ 'th bit for $\mu \in Z_{0, N_B-1}$. A bit string \vec{a} has a decimal representation $d(\vec{a}) = \sum_{\mu=0}^{N_B-1} 2^\mu a_\mu$. For $\beta \in Z_{0, N_B-1}$, we will use $\vec{u}(\beta)$ to denote the β 'th standard unit vector—i.e., the vector with bit value of 1 at bit position β and bit value of zero at all other bit positions.

The set $Bool^{N_B}$ can be ordered in the standard way, the *dictionary ordering*. For example, the dictionary ordering of $Bool^3$ is

$$(000, 001, 010, 011, 100, 101, 110, 111). \quad (2a.1)$$

Other useful orderings of $Bool^{N_B}$ are the so called *Gray codes*[17], named, not after the color, but after an actual person named Gray. In Gray codes, the next bit string may only differ from its predecessor in the value of a single bit. Clearly, this condition does not specify a unique ordering of $Bool^{N_B}$, so there is more than one Gray code of $Bool^{N_B}$. Henceforth, we will refer to a Gray code as a *lazy ordering*, because, as we step from any \vec{b} to the next, we act “lazily”, flipping only one bit instead of many. An example of a lazy ordering of $Bool^3$ is

$$(000, 100, 110, 010, 011, 111, 101, 001) . \quad (2a.2)$$

We define the single-qubit states $|0\rangle$ and $|1\rangle$ by

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \quad (2a.3)$$

If $\vec{a} \in Bool^{N_B}$, we define the N_B -qubit state $|\vec{a}\rangle$ as the following tensor product

$$|\vec{a}\rangle = |a_{N_B-1}\rangle \otimes \dots \otimes |a_1\rangle \otimes |a_0\rangle . \quad (2a.4)$$

For example,

$$|01\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} . \quad (2a.5)$$

I_r will represent the r dimensional unit matrix. Suppose $\beta \in Z_{0, N_B-1}$ and M is any 2×2 matrix. We define $M(\beta)$ by

$$M(\beta) = I_2 \otimes \dots \otimes I_2 \otimes M \otimes I_2 \otimes \dots \otimes I_2 , \quad (2a.6)$$

where the matrix M on the right side is located at bit position β in the tensor product of N_B 2×2 matrices. The numbers that label bit positions in the tensor product increase from right to left (\leftarrow), and the rightmost bit is taken to be at position 0.

For any pair of same-sized square matrices A and B such that $\det(B) \neq 0$, one can define $B^A = \exp(A \ln B)$ using the Taylor series of $\ln(\cdot)$ about the identity matrix and the Taylor series of $\exp(\cdot)$ about zero. This gives a definition for objects such as $M_1(\beta_1)^{M_2(\beta_2)}$, where M_1 and M_2 are 2×2 matrices, $\det(M_1) \neq 0$ and $\beta_1, \beta_2 \in Z_{0, N_B-1}$.

For any two same-sized square matrices A and B , we define the o-dot product \odot by $A \odot B = ABA^\dagger$, where A^\dagger is the Hermitian conjugate of A .

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ will represent the vector of Pauli matrices, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \quad (2a.7)$$

A *qubit rotation* is defined as any matrix of the form $\exp[i\vec{\theta} \cdot \vec{\sigma}(\beta)]$, where $\beta \in Z_{0, N_B-1}$ and $\vec{\theta}$ is a real 3-dimensional vector.

2(b) Projection Operators

Consider a single qubit first.

The number operator n of the qubit is defined by

$$n = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1 - \sigma_z}{2} . \quad (2b.1)$$

Note that

$$n|0\rangle = 0|0\rangle = 0 \quad , \quad n|1\rangle = 1|1\rangle . \quad (2b.2)$$

We will often use \bar{n} as shorthand for

$$\bar{n} = 1 - n = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1 + \sigma_z}{2} . \quad (2b.3)$$

Define P_0 and P_1 by

$$P_0 = \bar{n} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad P_1 = n = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} . \quad (2b.4)$$

P_0 and P_1 are orthogonal projectors and they add to one:

$$P_a P_b = \delta(a, b) P_b \quad \text{for } a, b \in Bool , \quad (2b.5)$$

$$P_0 + P_1 = I_2 . \quad (2b.6)$$

Now consider N_B bits instead of just one.

For $\beta \in Z_{0, N_B-1}$, we define $P_0(\beta)$, $P_1(\beta)$, $n(\beta)$ and $\bar{n}(\beta)$ according to Eq.(2a.6).[18]

For $\vec{a} \in Bool^{N_B}$, let

$$P_{\vec{a}} = P_{a_{N_B-1}} \otimes \cdots \otimes P_{a_2} \otimes P_{a_1} \otimes P_{a_0} . \quad (2b.7)$$

For example, with 2 bits we have

$$P_{00} = P_0 \otimes P_0 = \text{diag}(1, 0, 0, 0) , \quad (2b.8a)$$

$$P_{01} = P_0 \otimes P_1 = \text{diag}(0, 1, 0, 0) , \quad (2b.8b)$$

$$P_{10} = P_1 \otimes P_0 = \text{diag}(0, 0, 1, 0) , \quad (2b.8c)$$

$$P_{11} = P_1 \otimes P_1 = \text{diag}(0, 0, 0, 1) . \quad (2b.8d)$$

Note that

$$P_{\vec{a}} P_{\vec{b}} = \delta(\vec{a}, \vec{b}) P_{\vec{b}} \quad \text{for } \vec{a}, \vec{b} \in Bool^{N_B} , \quad (2b.9)$$

$$\sum_{\vec{a} \in Bool^{N_B}} P_{\vec{a}} = I_2 \otimes I_2 \otimes \cdots \otimes I_2 = I_{2^{N_B}} . \quad (2b.10)$$

For $r \geq 1$, suppose P_1, P_2, \dots, P_r are orthogonal projection operators (i.e., $P_i P_j = \delta(i, j) P_j$), and $\alpha_1, \alpha_2, \dots, \alpha_r$ are complex numbers. Then it is easy to show by Taylor expansion that

$$\exp\left(\sum_{i=1}^r \alpha_i P_i\right) = \sum_{i=1}^r \exp(\alpha_i) P_i + \left(1 - \sum_{i=1}^r P_i\right). \quad (2b.11)$$

In other words, one can “pull out” the summation sign from the argument of the exponential, but only if one adds a compensating term $1 - \sum_i P_i$ so that both sides of the equation agree when all the α_i 's are zero.

For any pair of 2×2 matrices τ_0, τ_1 , define

$$\tau_{\vec{b}} = \tau_{b_{N_B-1}} \otimes \dots \otimes \tau_{b_1} \otimes \tau_{b_0}, \quad (2b.12)$$

where $\vec{b} \in \text{Bool}^{N_B}$. Now let

$$\mathcal{B}(\tau_1, \tau_0) = \{\tau_{\vec{b}} | \vec{b} \in \text{Bool}^{N_B}\}. \quad (2b.13)$$

The set of $N_S \times N_S$ diagonal complex matrices is a vector space and $\mathcal{B}(P_1, P_0) = \mathcal{B}(n, \bar{n})$ is a basis for it. $\mathcal{B}(\sigma_z, I_2)$ and $\mathcal{B}(n, I_2)$ are also bases for it.

2(c) Sylvester-Hadamard Matrices

The N_B -bit Sylvester-Hadamard matrix[19] H_{N_B} is defined by:

$$H_1 = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & -1 \end{array}, \quad (2c.1a)$$

$$H_2 = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 1 & 1 & 1 \\ 01 & 1 & -1 & 1 & -1 \\ 10 & 1 & 1 & -1 & -1 \\ 11 & 1 & -1 & -1 & 1 \end{array}, \quad (2c.1b)$$

$$H_{r+1} = H_1 \otimes H_r, \quad (2c.1c)$$

for any integer $r \geq 1$. We will often use a plain H to represent H_1 . In Eqs.(2c.1), we have labelled the rows and columns with binary numbers in increasing dictionary order. From Eqs.(2c.1), one can show that the entry of H_{N_B} at row $\vec{a} \in \text{Bool}^{N_B}$ and column $\vec{b} \in \text{Bool}^{N_B}$ is given by

$$(H_{N_B})_{\vec{a}, \vec{b}} = (-1)^{\vec{a} \cdot \vec{b}}, \quad (2c.2)$$

where $\vec{a} \cdot \vec{b} = \sum_{\mu=0}^{N_B-1} a_{\mu} b_{\mu}$. It is easy to check that

$$H_{N_B}^T = H_{N_B} , \quad (2c.3)$$

$$H_{N_B}^2 = N_S I_{N_S} . \quad (2c.4)$$

In other words, H_{N_B} is a symmetric matrix, and the inverse of H_{N_B} equals H_{N_B} divided by however many rows it has.

If $H(\beta)$ for $\beta \in Z_{0, N_B-1}$ is defined according to Eq.(2a.6), then Eqs.(2c.1) imply that

$$H_{N_B} = H(N_B - 1) \dots H(2)H(1)H(0) . \quad (2c.5)$$

The $H(\beta)$'s on the right side of the last equation commute so also

$$H_{N_B} = H(0)H(1)H(2) \dots H(N_B - 1) . \quad (2c.6)$$

2(d) Discrete Fourier Transform

The N_B bit Discrete Fourier Transform (DFT) matrix F_{N_B} is defined by

$$(F_{N_B})_{a,b} = \frac{1}{\sqrt{N_S}} \omega^{ab} , \quad (2d.1)$$

where $a, b \in Z_{0, N_S-1}$ and

$$\omega = e^{i \frac{2\pi}{N_S}} . \quad (2d.2)$$

Note that F_{N_B} is a symmetric matrix. It is easy to show that it is also a unitary matrix. If \tilde{v} and v are complex N_S dimensional vectors such that $\tilde{v} = F_{N_B} v$, then we call \tilde{v} the DFT of v .

Calculating \tilde{v} the naive way, by multiplying v by F_{N_B} , would take $\text{Order}(N_S^2)$ classical elementary operations (complex multiplications mostly). Instead, it is possible to calculate \tilde{v} from v in $\text{Order}(N_S \ln N_S)$ classical elementary operations using the well known[2] Fast Fourier Transform (FFT) algorithm. Ref.[3] was the first to express this algorithm as a product of matrices each of which acts on at most 2 bits at a time. This way of expressing it is often called the ‘‘quantum FFT algorithm’’ because it is ideal for quantum computation. Ref.[3] showed (See Appendix B) that

$$F_{N_B} = \frac{1}{\sqrt{N_S}} H(N_B - 1) \dots \Delta(2)H(2)\Delta(1)H(1)\Delta(0)H(0)P_{BR} , \quad (2d.3)$$

where $H(\alpha)$ is the 1-bit Hadamard matrix operating on bit $\alpha \in Z_{0, N_B-1}$, P_{BR} is the bit reversal matrix for N_B bits, and

$$\Delta(\beta) = \Delta(\beta + 1, \beta)\Delta(\beta + 2, \beta) \dots \Delta(N_B - 1, \beta) , \quad (2d.4)$$

where

$$\Delta(\alpha, \beta) = \exp[i\phi_{|\alpha-\beta|+1}n(\alpha)n(\beta)] , \quad (2d.5)$$

$$\phi_\gamma = \frac{2\pi}{2^\gamma} . \quad (2d.6)$$

Thus, $\Delta(\alpha, \beta)$ is a diagonal matrix whose diagonal entries are either 1 or a phase factor. For example, for $N_B = 3$,

$$n(0)n(2)|a_2, a_1, a_0\rangle = \begin{cases} 1|a_2, a_1, a_0\rangle & \text{if } a_2 = a_0 = 1 \\ 0 & \text{otherwise} \end{cases} , \quad (2d.7)$$

so

$$\Delta(0, 2) = e^{i\phi_3 n(0)n(2)} = \text{diag}(\overset{000}{1}, \overset{001}{1}, \overset{010}{1}, \overset{011}{1}, \overset{100}{1}, \overset{101}{e^{i\phi_3}}, \overset{110}{1}, \overset{111}{e^{i\phi_3}}) . \quad (2d.8)$$

For $N_B = 3$, reversing the bits of the numbers contained in $Z_{0,7}$ exchanges $1 = d(001)$ with $4 = d(100)$ and $3 = d(011)$ with $6 = d(110)$, and it leaves all other numbers in $Z_{0,7}$ the same. Thus, for $N_B = 3$, P_{BR} is the 8×8 permutation matrix which corresponds to the following product of transpositions: $(1,4)(3,6)$.

Note that F_{N_B} , $H(\alpha)$, $\Delta(\alpha)$ and P_{BR} are all symmetric matrices. Hence, taking the transpose of both sides of Eq.(2d.3), one gets

$$F_{N_B} = \frac{1}{\sqrt{N_S}} P_{BR} H(0) \Delta(0) H(1) \Delta(1) H(2) \Delta(2) \dots H(N_B - 1) . \quad (2d.9)$$

Both the last equation and Eq.(2d.3) are called the quantum FFT algorithm.

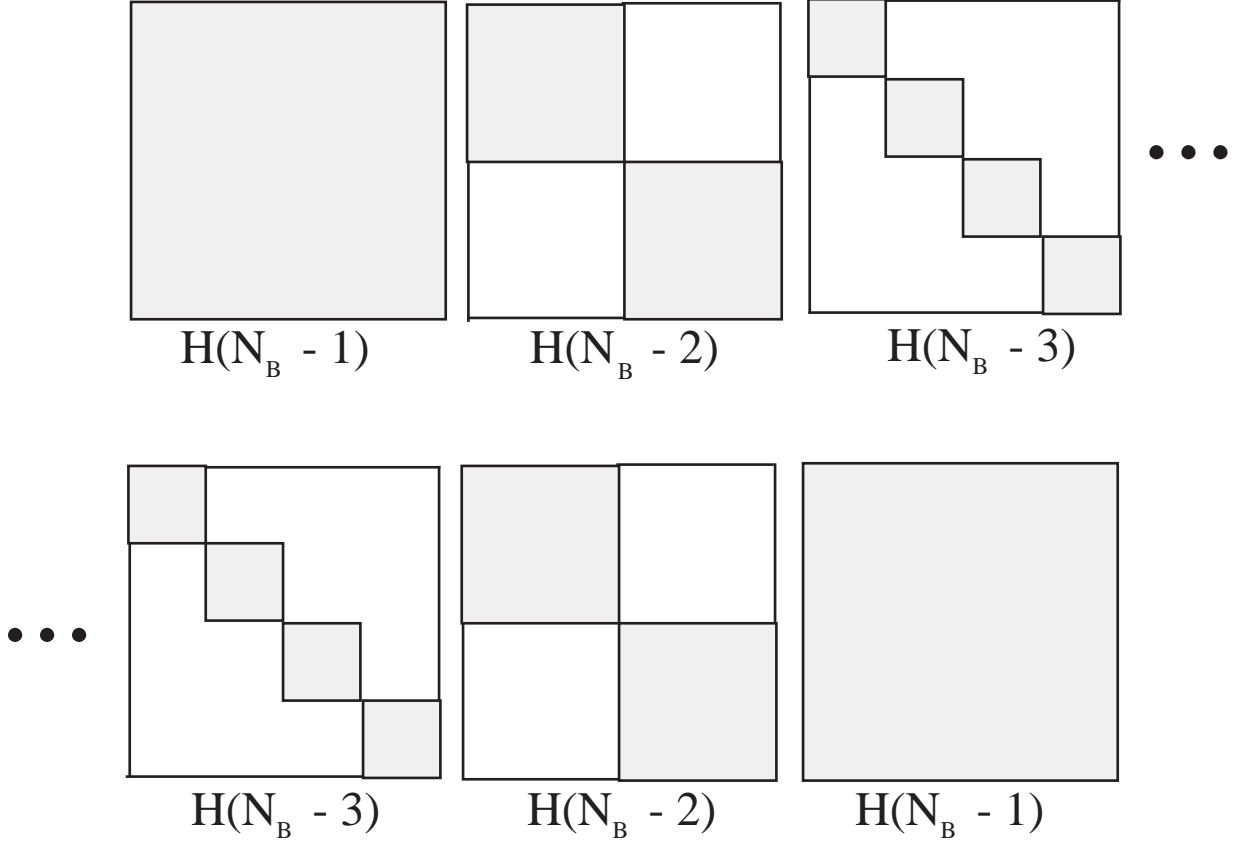


Fig.2 Pictorial representation of quantum FFT algorithm. For simplicity, we only show the Hadamard matrices. White (unshaded) regions inside a matrix represent zero entries.

Fig.2 is a pictorial representation of Eqs.(2d.3) and (2d.9). For any pair of 2×2 complex matrices M, M' and for $\beta, \beta' \in Z_{0, N_B-1}$, we will say that $M(\beta)$ is “more diagonal” than $M'(\beta)$ if $\beta < \beta'$. $M(0)$ is a direct sum of 2×2 matrices. $M(1)$ is a direct sum of 4×4 matrices so it is less diagonal than $M(0)$, etc. The matrices $\Delta(\alpha)$ that occur in Eqs.(2d.3) and (2d.9) are truly diagonal but the matrices $H(\alpha)$ aren't. In Eq.(2d.3) (ditto, Eq.(2d.9)) the $H(\alpha)$ matrices become less diagonal (ditto, more diagonal) as one goes from right to left.

The fact that the non-diagonal matrices $H(\alpha)$ are real and the diagonal matrices $\Delta(\alpha)$ are complex is what gives the FFT algorithm its speed advantage in classical computation. Of course, segregating real and complex operations has no advantage in quantum computation, where complex numbers are very natural. The charm of the FFT algorithm in quantum computation is that it can be expressed as a short SEO, namely Eqs.(2d.3) and (2d.9).

2(e) Permutations

Subsequent sections will use the following very basic facts about permutations. For more details, see, for example, Ref.[20].

A *permutation* is a 1-1 onto map from a finite set X onto itself. The set of permutations on set X is a group if group multiplication is taken to be function composition. S_n , the *symmetric group in n letters*, is defined as the group of all permutations on any set X with n elements. If $X = Z_{1,n}$, then a permutation G which maps $i \in X$ to $a_i \in X$ (where $i \neq j$ implies $a_i \neq a_j$) can be represented by a matrix with entries

$$(G)_{j,i} = \delta(a_i, j) , \quad (2e.1)$$

for all $i, j \in X$. Note that all entries in any given row or column equal zero except for one entry which equals one. Hence, the rows of G are orthonormal and G is an orthogonal matrix ($G^T G = G G^T = 1$). An alternative notation for G is

$$G = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix} . \quad (2e.2)$$

The product of two symbols of the type shown in Eq.(2e.2) is defined by function composition. For example,

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ a_1 & a_2 & a_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ b_1 & b_2 & b_3 \end{pmatrix} . \quad (2e.3)$$

Note how we have applied the permutations on the left side of the equation from right to left (\leftarrow). (Careful: Some authors apply them in the opposite direction (\rightarrow)). A cycle is a special type of permutation. If $G \in S_n$ maps $a_1 \rightarrow a_2, a_2 \rightarrow a_3, \dots, a_{r-1} \rightarrow a_r, a_r \rightarrow a_1$, where $i \neq j$ implies $a_i \neq a_j$ and $r \leq n$, then we call G a *cycle*. G may be represented as in Eqs.(2e.1) and (2e.2). Another way to represent it is by

$$G = (a_1, a_2, a_3, \dots, a_r) . \quad (2e.4)$$

(Careful: some people write $(a_r, \dots, a_3, a_2, a_1)$ instead.) We say that the cycle of Eq.(2e.4) has *length* r . Cycles of length 1 are just the identity map. A cycle of length 2 is called a *transposition*. The product of two cycles need not be another cycle. For example,

$$(2, 1, 5)(1, 4, 5, 6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 6 & 5 \end{pmatrix} \quad (2e.5)$$

cannot be expressed as a single cycle. Any permutation can be written as a product of cycles. For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 6 & 5 \end{pmatrix} = (5, 6)(1, 4, 2) . \quad (2e.6)$$

The cycles on the right side of Eq.(2e.6) are *disjoint*; i.e., they have no elements in common. Disjoint cycles commute. Any cycle can be expressed as a product of transpositions (assuming a group with ≥ 2 elements), by using identities such as:

$$(a_1, a_2, \dots, a_n) = (a_1, a_2)(a_2, a_3) \cdots (a_{n-1}, a_n) , \quad (2e.7)$$

$$(a_1, a_2, \dots, a_n) = (a_1, a_n) \cdots (a_1, a_3)(a_1, a_2) . \quad (2e.8)$$

Another useful identity is

$$(a, b) = (a, p)(p, b)(a, p) . \quad (2e.9)$$

This last identity can be applied repeatedly. For example, applied twice, it gives

$$(a, b) = (a, p_1)(p_1, b)(a, p_1) = (a, p_1)(p_1, p_2)(p_2, b)(p_1, p_2)(a, p_1) . \quad (2e.10)$$

Since any permutation equals a product of cycles, and each of those cycles can be expressed as a product of transpositions, all permutations can be expressed as a product of transpositions (assuming a group with ≥ 2 elements). The decomposition of a permutation into transpositions is not unique. However, the number of transpositions whose product equals a given permutation is always either even or odd. An *even* (ditto, *odd*) *permutation* is defined as one which equals an even (ditto, odd) number of transpositions.

3. State Permutations that Act on Two Bits

The goal of this paper is to reduce any unitary matrix into a product of qubit rotations and controlled-nots (c-nots). A qubit rotation acts on a single qubit at a time. This section will discuss gates such as c-nots that are state permutations that act on two bits at a time.

3(a) $N_B = 2$

Consider first the case when there are only 2 bits. Then there are four possible states—00, 01, 10, 11. With these 4 states, one can build 6 distinct transpositions:

$$(00, 01) = \begin{bmatrix} \sigma_x & \\ & I_2 \end{bmatrix} = P_0 \otimes \sigma_x + P_1 \otimes I_2 = \sigma_x(0)^{\bar{n}(1)} , \quad (3a.1a)$$

$$(00, 10) = \begin{bmatrix} P_1 & P_0 \\ P_0 & P_1 \end{bmatrix} = I_2 \otimes P_1 + \sigma_x \otimes P_0 = \sigma_x(1)^{\bar{n}(0)} , \quad (3a.1b)$$

$$(00, 11) = \begin{bmatrix} & & 1 \\ & I_2 & \\ 1 & & \end{bmatrix}, \quad (3a.1c)$$

$$(01, 10) = \begin{bmatrix} 1 & & \\ & \sigma_x & \\ & & 1 \end{bmatrix}, \quad (3a.1d)$$

$$(01, 11) = \begin{bmatrix} P_0 & P_1 \\ P_1 & P_0 \end{bmatrix} = I_2 \otimes P_0 + \sigma_x \otimes P_1 = \sigma_x(1)^{n(0)}, \quad (3a.1e)$$

$$(10, 11) = \begin{bmatrix} I_2 & \\ & \sigma_x \end{bmatrix} = P_0 \otimes I_2 + P_1 \otimes \sigma_x = \sigma_x(0)^{n(1)}, \quad (3a.1f)$$

where matrix entries left blank should be interpreted as zero. The rows and columns of the above matrices are labelled by binary numbers in increasing dictionary order (as in Eq.(2c.1b) for H_2). Note that the 4 transpositions Eqs.(3a.1)(a,b,e,f) change only one bit value. We will call them *controlled nots (c-nots)*. The other 2 transpositions Eqs.(3a.1)(c,d) change both bit values. We will call (00, 11) the *Twin-to-twin-er* and (01, 10) the *Exchanger*. Expressions such as $\sigma_x(\beta)^{n(\alpha)}$ where $\alpha \neq \beta$ are a special case of $M_1(\beta_1)^{M_2(\beta_2)}$, which was defined in Section 2a. $\sigma_x(\beta)^{n(\alpha)}$ equals $\sigma_x(\beta)$ when it acts on a state for which $n(\alpha) = 1$, whereas it equals 1 if $n(\alpha) = 0$. α is called the *control bit* and β the *flipper bit*.

Exchanger[21] has four possible representations as a product of c-nots:

$$(01, 10) = (01, 00)(00, 10)(01, 00) = \sigma_x(0)^{\bar{n}(1)}\sigma_x(1)^{\bar{n}(0)}\sigma_x(0)^{\bar{n}(1)}, \quad (3a.2a)$$

$$(01, 10) = (10, 11)(11, 01)(10, 11) = \sigma_x(0)^{n(1)}\sigma_x(1)^{n(0)}\sigma_x(0)^{n(1)}, \quad (3a.2b)$$

$$(01, 10) = (10, 00)(00, 01)(10, 00) = \sigma_x(1)^{\bar{n}(0)}\sigma_x(0)^{\bar{n}(1)}\sigma_x(1)^{\bar{n}(0)}, \quad (3a.2c)$$

$$(01, 10) = (01, 11)(11, 10)(01, 11) = \sigma_x(1)^{n(0)}\sigma_x(0)^{n(1)}\sigma_x(1)^{n(0)}. \quad (3a.2d)$$

Note that one can go from Eq.(3a.2a) to (3a.2b) by exchanging n and \bar{n} ; from Eq.(3a.2a) to (3a.2c) by exchanging bit positions 0 and 1; from Eq.(3a.2a) to (3a.2d) by doing both, exchanging n and \bar{n} and exchanging bit positions 0 and 1. We will often represent Exchanger by $E(0, 1)$. It is easy to show that

$$E^T(0, 1) = E(0, 1) = E^{-1}(0, 1) , \quad (3a.3a)$$

$$E(0, 1) = E(1, 0) , \quad (3a.3b)$$

$$E^2(0, 1) = 1 . \quad (3a.3c)$$

Furthermore, if X and Y are two arbitrary 2×2 matrices, then, by using the matrix representation Eq.(3a.1d) of Exchanger, one can show that

$$E(1, 0) \odot (X \otimes Y) = Y \otimes X . \quad (3a.4)$$

Thus, Exchanger exchanges the position of matrices X and Y in the tensor product.

Twin-to-twin-er also has 4 possible representations as a product of c-nots. One is

$$(00, 11) = (00, 01)(01, 11)(00, 01) = \sigma_x(0)^{\bar{n}(1)}\sigma_x(1)^{n(0)}\sigma_x(0)^{\bar{n}(1)} . \quad (3a.5)$$

As with Exchanger, the other 3 representations are obtained by exchanging: (1) n and \bar{n} , (2) bit positions 0 and 1, (3) both.

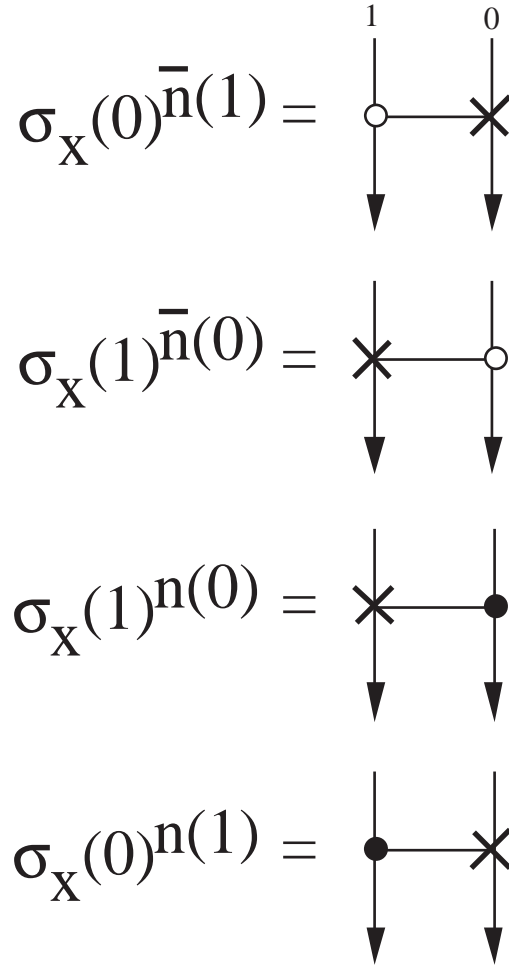


Fig.3 Circuit symbols for the 4 different types of c-nots.

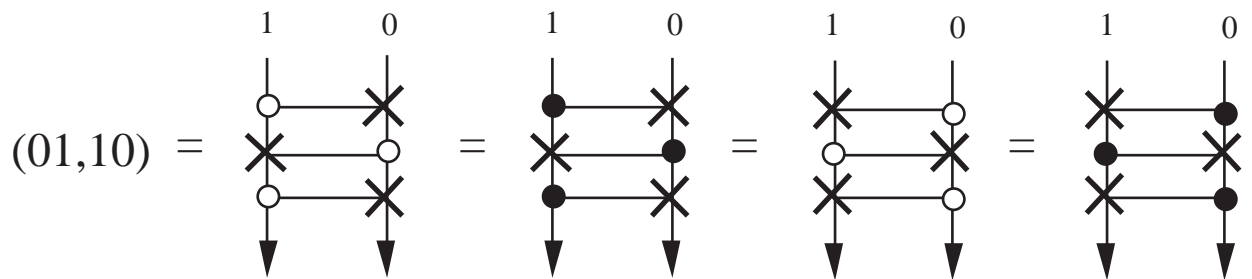


Fig.4 Four equivalent circuit diagrams for Exchanger.

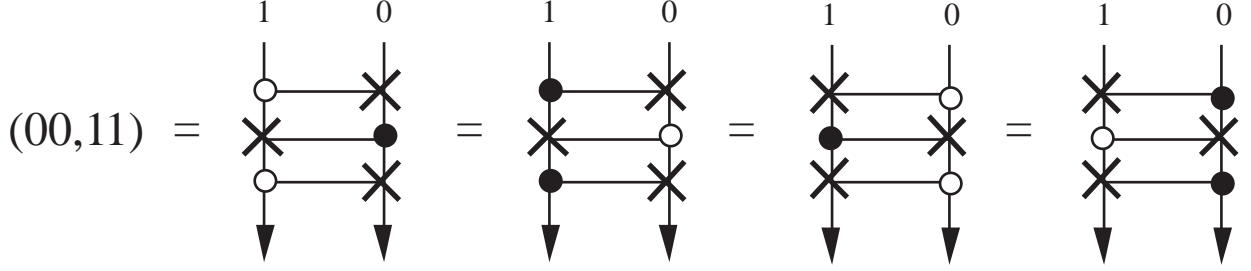


Fig.5 Four equivalent circuit diagrams for Twin-to-twin-er.

Figures 3, 4 and 5 give a diagrammatic representation of the 6 possible transpositions of states for $N_B = 2$.

3(b) Any $N_B \geq 2$

Suppose $a_1, b_1, a_2, b_2 \in Bool$ and $\alpha, \beta \in Z_{0, N_B-1}$ such that $\alpha \neq \beta$. We define

$$(a_1 b_1, a_2 b_2)_{\alpha, \beta} = \prod_{(\Lambda, \Lambda', \Lambda'') \in Bool^{N_B-2}} (\Lambda a_1 \Lambda' b_1 \Lambda'', \Lambda a_2 \Lambda' b_2 \Lambda''), \quad (3b.1)$$

where on the right side, a_1, a_2 are located at bit position α , and b_1, b_2 are located at bit position β . (Note that the transpositions on the right side of Eq.(3b.1) are disjoint so they commute.) For example, for $N_B = 3$,

$$\sigma_x(0)^{n(1)} = (10, 11)_{1,0} = \prod_{a \in Bool} (a10, a11) = (010, 011)(110, 111). \quad (3b.2)$$

Clearly, any permutation of states with N_B bits that acts on only 2 bits (i.e., Exchanger, Twin-to-twin-er, and all c-nots) can be represented by $(a_1 b_1, a_2 b_2)_{\alpha, \beta}$.

For $\alpha, \beta \in Z_{0, N_B-1}$ such that $\alpha \neq \beta$, let $E(\alpha, \beta)$ represent Exchanger:

$$E(\alpha, \beta) = (01, 10)_{\alpha, \beta}. \quad (3b.3)$$

As in the $N_B = 2$ case, $E(\alpha, \beta)$ can be expressed as a product of c-nots in 4 different ways. One way is

$$E(\alpha, \beta) = \sigma_x(\alpha)^{n(\beta)} \sigma_x(\beta)^{n(\alpha)} \sigma_x(\alpha)^{n(\beta)}. \quad (3b.4)$$

The other 3 ways are obtained by exchanging: (1) n and \bar{n} , (2) bit positions α and β , (3) both. Again as in the $N_B = 2$ case,

$$E^T(\alpha, \beta) = E(\alpha, \beta) = E^{-1}(\alpha, \beta), \quad (3b.5a)$$

$$E(\alpha, \beta) = E(\beta, \alpha), \quad (3b.5b)$$

$$E^2(\alpha, \beta) = 1. \quad (3b.5c)$$

Furthermore, if X and Y are two arbitrary 2×2 matrices and $\alpha, \beta \in Z_{0, N_B-1}$ such that $\alpha \neq \beta$, then

$$E(\alpha, \beta) \odot [X(\alpha)Y(\beta)] = X(\beta)Y(\alpha). \quad (3b.6)$$

Equation (3b.6) is an extremely useful result. It says that $E(\alpha, \beta)$ is a transposition of bit positions. Furthermore, the $E(\alpha, \beta)$ generate the group of $N_B!$ permutations of bit positions. (Careful: this is not the same as the group of $(2^{N_B})!$ permutations of states with N_B bits.)

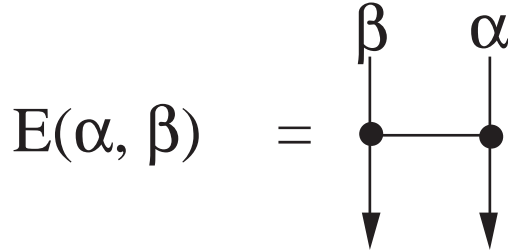


Fig.6 Circuit symbol for Exchanger.

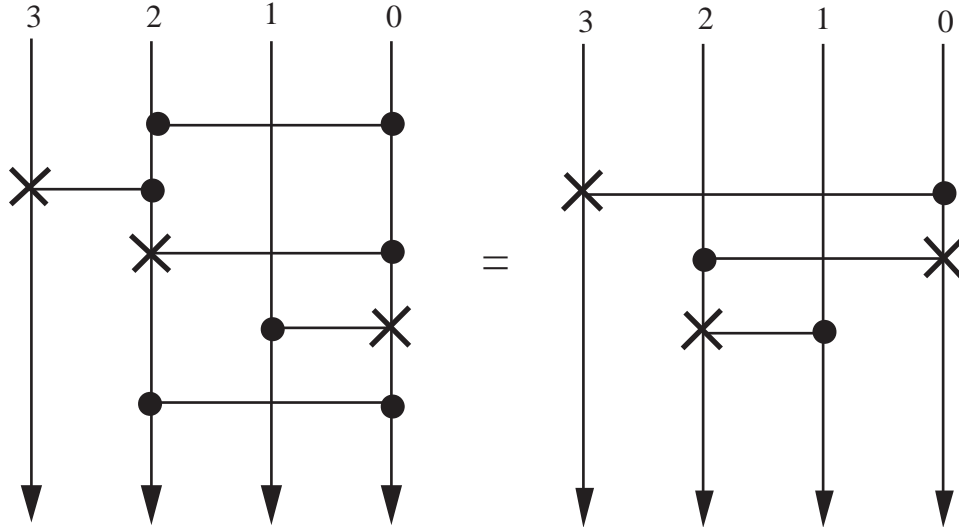


Fig.7 Circuit diagram for Eq.(3b.7).

An example of how one can use Eq.(3b.6) is

$$E(2, 0) \odot [\sigma_x(0)^{n(1)} \sigma_x(2)^{n(0)} \sigma_x(3)^{n(2)}] = \sigma_x(2)^{n(1)} \sigma_x(0)^{n(2)} \sigma_x(3)^{n(0)}. \quad (3b.7)$$

Figure 6 gives a convenient way of representing $E(\alpha, \beta)$ diagrammatically. Using this symbol, the example of Eq.(3b.7) can be represented by Fig.7.

Certain experimental implementations of a quantum computer might only allow nearest-neighbor bit interactions. Exchanger can be used to express any elementary operation between 2 non-nearest-neighbor bits as a SEO that contains only nearest-neighbor interactions. For example, if $A(3, 0)$ is an operator that acts on bits 3 and 0, then

$$A(3, 0) = [E(3, 2)E(2, 1)] \odot A(1, 0) . \quad (3b.8)$$

Of course, identities that are true for a general transposition are also true for $E(\alpha, \beta)$. For example,

$$(2, 0) = (2, 1)(1, 0)(2, 1) . \quad (3b.9)$$

Therefore,

$$E(2, 0) = E(2, 1)E(1, 0)E(2, 1) . \quad (3b.10)$$

4. Decomposing Central Matrix into SEO

In Section 1(c), we gave only a partial description of our algorithm. In this section, we complete that description by showing how to decompose each of the 3 possible kinds of central matrices into a SEO.

4(a) When Central Matrix is a Single D Matrix

D matrices are defined by Eqs.(1b.2). They can be expressed in terms of projection operators as follows:

$$D = \sum_{\vec{a} \in \text{Bool}^{N_B-1}} \exp(i\phi_{\vec{a}}\sigma_y) \otimes P_{\vec{a}} , \quad (4a.1)$$

where the $\phi_{\vec{a}}$ are real numbers. Note that in Eq.(4a.1), \vec{a} has $N_B - 1$ components instead of the full N_B . Using the identity Eq.(2b.11), one gets

$$D = \exp \left(i \sum_{\vec{a} \in \text{Bool}^{N_B-1}} \phi_{\vec{a}} \sigma_y \otimes P_{\vec{a}} \right) . \quad (4a.2)$$

Now define new angles $\theta_{\vec{b}}$ by

$$\phi_{\vec{a}} = \sum_{\vec{b} \in \text{Bool}^{N_B-1}} (-1)^{\vec{a} \cdot \vec{b}} \theta_{\vec{b}} . \quad (4a.3)$$

Suppose $\vec{\phi}$ (ditto, $\vec{\theta}$) is a column vector whose components are the numbers $\phi_{\vec{a}}$ (ditto, $\theta_{\vec{a}}$) arranged in order of increasing \vec{a} . Then Eq.(4a.3) is equivalent to

$$\vec{\phi} = H_{N_B-1} \vec{\theta} . \quad (4a.4)$$

This is easily inverted to

$$\vec{\theta} = \frac{1}{2^{N_B-1}} H_{N_B-1} \vec{\phi}. \quad (4a.5)$$

Let $A_{\vec{b}}$ for $\vec{b} \in \text{Bool}^{N_B-1}$ be defined by

$$A_{\vec{b}} = \exp \left(i\theta_{\vec{b}} \sigma_y \otimes \sum_{\vec{a} \in \text{Bool}^{N_B-1}} (-1)^{\vec{a} \cdot \vec{b}} P_{\vec{a}} \right). \quad (4a.6)$$

Then D can be written as

$$D = \prod_{\vec{b} \in \text{Bool}^{N_B-1}} A_{\vec{b}}. \quad (4a.7)$$

Note that the $A_{\vec{b}}$ operators on the right side commute so the order in which they are multiplied is irrelevant. Next we establish 2 useful identities: If $\beta \in Z_{0, N_B-1}$ and $\vec{u}(\beta) \in \text{Bool}^{N_B}$ is the β 'th standard unit vector, then

$$\begin{aligned} & \sum_{\vec{a} \in \text{Bool}^{N_B}} (-1)^{\vec{a} \cdot \vec{u}(\beta)} P_{\vec{a}} \\ &= I_2 \otimes \cdots \otimes I_2 \otimes \left[\sum_{\alpha \in \text{Bool}} (-1)^{\alpha \beta} P_{\alpha \beta} \right] \otimes I_2 \otimes \cdots \otimes I_2 \\ &= P_0(\beta) - P_1(\beta) \\ &= \sigma_z(\beta) \end{aligned} \quad (4a.8)$$

If $\beta, \alpha \in Z_{0, N_B-1}$ and $\alpha \neq \beta$, then

$$\begin{aligned} & \sigma_x(\beta)^{n(\alpha)} \odot \sigma_y(\beta) \\ &= [\sigma_x(\beta) P_1(\alpha) + P_0(\alpha)] \odot \sigma_y(\beta) \\ &= \sigma_y(\beta) [-P_1(\alpha) + P_0(\alpha)] \\ &= \sigma_y(\beta) \sigma_z(\alpha) \end{aligned} \quad (4a.9)$$

Now we are ready to express $A_{\vec{b}}$ in terms of elementary operators. For any $\vec{b} \in \text{Bool}^{N_B-1}$, we can write

$$\vec{b} = \sum_{j=0}^{r-1} \vec{u}(\beta_j), \quad (4a.10)$$

where

$$N_B - 2 \geq \beta_{r-1} > \cdots > \beta_1 > \beta_0 \geq 0. \quad (4a.11)$$

In other words, \vec{b} has bit value of 1 at bit positions β_j . At all other bit positions, \vec{b} has bit value of 0. r is the number of bits in \vec{b} whose value is 1. When $\vec{b} = 0$, r equals 0. By Eq.(4a.8)

$$A_{\vec{b}} = \exp \left(i\theta_{\vec{b}} \sigma_y (N_B - 1) \prod_{j=0}^{r-1} \sigma_z(\beta_j) \right). \quad (4a.12)$$

By Eq.(4a.9), if $r \geq 1$,

$$\begin{aligned} & [\sigma_x(N_B - 1)^{n(\beta_{r-1})} \dots \sigma_x(N_B - 1)^{n(\beta_1)} \sigma_x(N_B - 1)^{n(\beta_0)}] \odot \sigma_y(N_B - 1) \\ & = \sigma_y(N_B - 1) \prod_{j=0}^{r-1} \sigma_z(\beta_j) \end{aligned} \quad (4a.13)$$

Thus, for $r \geq 0$,

$$A_{\vec{b}} = [\sigma_x(N_B - 1)^{n(\beta_{r-1})} \dots \sigma_x(N_B - 1)^{n(\beta_1)} \sigma_x(N_B - 1)^{n(\beta_0)}] \odot \exp[i\theta_{\vec{b}} \sigma_y(N_B - 1)] , \quad (4a.14)$$

where if $r = 0$, the expression to the left of \odot is defined to be 1. There are other ways of decomposing $A_{\vec{b}}$ into a SEO. For example, using the above method, one can also show that

$$A_{\vec{b}} = [\sigma_x(\beta_{r-2})^{n(\beta_{r-1})} \dots \sigma_x(\beta_1)^{n(\beta_2)} \sigma_x(\beta_0)^{n(\beta_1)} \sigma_x(N_B - 1)^{n(\beta_0)}] \odot \exp[i\theta_{\vec{b}} \sigma_y(N_B - 1)] . \quad (4a.15)$$

In conclusion, we have shown how to decompose a D matrix into a SEO. For example, suppose $N_B = 3$. Then

$$D = \sum_{a,b \in \text{Bool}} \exp(i\phi_{ab} \sigma_y) \otimes P_a \otimes P_b . \quad (4a.16)$$

Define $\vec{\theta}$ by

$$\vec{\theta} = \frac{1}{4} H_2 \vec{\phi} , \quad (4a.17)$$

and $\Gamma(\cdot)$ by

$$\Gamma(\theta) = \exp(i\theta \sigma_y) \otimes I_2 \otimes I_2 . \quad (4a.18)$$

Then

$$D = A_{00} A_{01} A_{10} A_{11} , \quad (4a.19)$$

where

$$A_{00} = \Gamma(\theta_{00}) , \quad (4a.20a)$$

$$A_{01} = \sigma_x(2)^{n(0)} \odot \Gamma(\theta_{01}) , \quad (4a.20b)$$

$$A_{10} = \sigma_x(2)^{n(1)} \odot \Gamma(\theta_{10}) , \quad (4a.20c)$$

$$A_{11} = [\sigma_x(2)^{n(1)} \sigma_x(2)^{n(0)}] \odot \Gamma(\theta_{11}) . \quad (4a.20d)$$

4(b) When Central Matrix is a Direct Sum of D Matrices

Consider first the case $N_B = 3$. Let $R(\phi) = \exp(i\sigma_y\phi)$. Previously we used the fact that any D matrix D can be expressed as

$$D = \sum_{a,b \in Bool} R(\phi''_{ab}) \otimes P_a \otimes P_b . \quad (4b.1)$$

But what if R were located at bit positions 0 or 1 instead of 2? The next two equations can be proven by expressing both sides of the equation as an 8×8 matrix.

$$D_0 \oplus D_1 = \sum_{a,b \in Bool} P_a \otimes R(\phi'_{ab}) \otimes P_b , \quad (4b.2)$$

$$D_{00} \oplus D_{01} \oplus D_{10} \oplus D_{11} = \sum_{a,b \in Bool} P_a \otimes P_b \otimes R(\phi_{ab}) , \quad (4b.3)$$

where the D_j and D_{ij} are D matrices. One can apply a string of Exchangers to move R in Eqs.(4b.2) and (4b.3) to any bit position. Thus,

$$D_0 \oplus D_1 = E(1,2) \odot \left(\sum_{a,b \in Bool} R(\phi'_{ab}) \otimes P_a \otimes P_b \right) , \quad (4b.4)$$

$$D_{00} \oplus D_{01} \oplus D_{10} \oplus D_{11} = [E(0,1)E(1,2)] \odot \left(\sum_{a,b \in Bool} R(\phi_{ab}) \otimes P_a \otimes P_b \right) . \quad (4b.5)$$

(Careful: $E(0,2) \neq E(0,1)E(1,2)$. $E(0,2)$ will change $\sum_{a,b} R(\phi_{ab}) \otimes P_a \otimes P_b$ to $\sum_{a,b} P_b \otimes P_a \otimes R(\phi_{ab})$, which is not the same as the right side of Eq.(4b.3)).

For general $N_B \geq 1$, if $\beta \in Z_{0,N_B-1}$ and

$$E = \begin{cases} 1 & \text{if } \beta = 0 \text{ or } N_B = 1 \\ E(N_B - \beta - 1, N_B - \beta) \cdots E(N_B - 3, N_B - 2)E(N_B - 2, N_B - 1) & \text{otherwise} \end{cases} , \quad (4b.6)$$

then a direct sum of 2^β D matrices can be expressed as

$$E \odot \left(\sum_{\vec{a} \in Bool^{N_B-1}} R(\phi_{\vec{a}}) \otimes P_{\vec{a}} \right) . \quad (4b.7)$$

It follows that if we want to decompose a direct sum of D matrices into a SEO, we can do so in 2 steps: (1) decompose into a SEO the D matrix that one obtains by moving the qubit rotation to bit position $N_B - 1$, (2) Replace each bit name in the decomposition by its “alias”. By alias we mean the new name assigned by the bit permutation E defined by Eq.(4b.6).

4(c) When Central Matrix is a Diagonal Unitary Matrix

Any diagonal unitary matrix Δ can be expressed as

$$\Delta = \sum_{\vec{a} \in \text{Bool}^{N_B}} \exp(i\phi_{\vec{a}}) P_{\vec{a}}, \quad (4c.1)$$

where the $\phi_{\vec{a}}$ are real numbers. Using the identity Eq.(2b.11) yields

$$\Delta = \exp \left(i \sum_{\vec{a} \in \text{Bool}^{N_B}} \phi_{\vec{a}} P_{\vec{a}} \right). \quad (4c.2)$$

Now define new angles $\theta_{\vec{b}}$ by

$$\phi_{\vec{a}} = \sum_{\vec{b} \in \text{Bool}^{N_B}} (-1)^{\vec{a} \cdot \vec{b}} \theta_{\vec{b}}. \quad (4c.3)$$

In terms of vectors,

$$\vec{\phi} = H_{N_B} \vec{\theta}, \quad (4c.4)$$

and

$$\vec{\theta} = \frac{1}{2^{N_B}} H_{N_B} \vec{\phi}. \quad (4c.5)$$

Let $A_{\vec{b}}$ for $\vec{b} \in \text{Bool}^{N_B}$ be defined by

$$A_{\vec{b}} = \exp \left(i\theta_{\vec{b}} \sum_{\vec{a} \in \text{Bool}^{N_B}} (-1)^{\vec{a} \cdot \vec{b}} P_{\vec{a}} \right). \quad (4c.6)$$

Then Δ can be written as

$$\Delta = \prod_{\vec{b} \in \text{Bool}^{N_B}} A_{\vec{b}}, \quad (4c.7)$$

where the $A_{\vec{b}}$ operators commute. For any $\vec{b} \in \text{Bool}^{N_B}$, we can write

$$\vec{b} = \sum_{j=0}^{r-1} \vec{u}(\beta_j), \quad (4c.8)$$

where

$$N_B - 1 \geq \beta_{r-1} > \cdots > \beta_1 > \beta_0 \geq 0. \quad (4c.9)$$

(Careful: Compare this with Eq.(4a.11). Now $\vec{b} \in \text{Bool}^{N_B}$ instead of Bool^{N_B-1} and β_{r-1} can be as large as $N_B - 1$ instead of $N_B - 2$.) One can show using the techniques of Section 4(a) that

$$A_{\vec{b}} = \begin{cases} \exp[i\theta_0] & \text{if } r = 0 \\ \exp[i\theta_{\vec{b}}\sigma_z(\beta_0)] & \text{if } r = 1 \\ [\sigma_x(\beta_0)^{n(\beta_{r-1})} \dots \sigma_x(\beta_0)^{n(\beta_2)} \sigma_x(\beta_0)^{n(\beta_1)}] \odot \exp[i\theta_{\vec{b}}\sigma_z(\beta_0)] & \text{if } r \geq 2 \end{cases} . \quad (4c.10)$$

As in Section 4(a), there are other ways of decomposing $A_{\vec{b}}$ into a SEO.

In conclusion, we have shown how to decompose a diagonal unitary matrix into a SEO. For example, suppose $N_B = 2$. Then

$$\Delta = \text{diag}(e^{i\phi_{00}}, e^{i\phi_{01}}, e^{i\phi_{10}}, e^{i\phi_{11}}) . \quad (4c.11)$$

Define $\vec{\theta}$ by

$$\vec{\theta} = \frac{1}{4} H_2 \vec{\phi} . \quad (4c.12)$$

By Eqs.(4c.7) and (4c.10),

$$\Delta = A_{00} A_{01} A_{10} A_{11} , \quad (4c.13)$$

where

$$A_{00} = \exp(i\theta_{00}) , \quad (4c.14a)$$

$$A_{01} = I_2 \otimes \exp(i\theta_{01}\sigma_z) , \quad (4c.14b)$$

$$A_{10} = \exp(i\theta_{10}\sigma_z) \otimes I_2 , \quad (4c.14c)$$

$$A_{11} = \sigma_x(0)^{n(1)} \odot [I_2 \otimes \exp(i\theta_{11}\sigma_z)] . \quad (4c.14d)$$

4(d) Comments

The use of a Hadamard transform in Section 4(a) may seem at first somewhat mysterious to the reader. Sorry. Here is some motivation for it.

We began Section 4(a) by noting with Eq.(4a.2) that any D matrix D can be expressed as $B = \exp[i\sigma_y(N_B - 1)]$ raised to some power which equals a linear combination of products of $P_0(\beta) = \bar{n}(\beta)$ and $P_1(\beta) = n(\beta)$, where $\beta \in Z_{0, N_B - 2}$. Using a Hadamard transform allowed us to re-express D as a product of $A_{\vec{b}}$'s . According to Eq.(4a.12), each $A_{\vec{b}}$ can be expressed as B raised to some power which equals a product of $\sigma_z(\beta)$, where $\beta \in Z_{0, N_B - 2}$. If in Eq.(4a.2) we had replaced each $n(\beta)$ and $\bar{n}(\beta)$ by $\frac{1}{2}[1 - \sigma_z(\beta)]$ and $\frac{1}{2}[1 + \sigma_z(\beta)]$, respectively, then we would have found the same thing that we found via a Hadamard transform—i.e., that D is expressible as a product of $A_{\vec{b}}$'s, each of which is expressible as B raised to some product of $\sigma_z(\beta)$'s.

For any 2×2 complex matrix B such that $\det(B) \neq 0$, we define a *controlled gate* g to be an operator of the form

$$g = B(\alpha)^{P_{b_0}(\beta_0)P_{b_1}(\beta_1)\dots P_{b_{r-1}}(\beta_{r-1})}, \quad (4d.1)$$

where $b_0, b_1, \dots, b_{r-1} \in Bool$, and $\beta_0, \beta_1, \dots, \beta_{r-1}, \alpha$ are distinct elements of Z_{0, N_B-1} . We call $\beta_0, \beta_1, \dots, \beta_{r-1}$ the *control bits* and α the *flipper bit*. An example of Eq.(4d.1) is when $B = e^{i\phi}$ for some real ϕ , in which case we call g a *controlled phase factor*. We already encountered controlled phase factors, with 2 controls, in the FFT algorithm. Two other examples of Eq.(4d.1) are when $B = \sigma_x(\alpha)$ and $B = \exp[i\vec{\theta} \cdot \vec{\sigma}(\alpha)]$ for some real 3-dimensional vector $\vec{\theta}$, in which cases we get a *c-not* (with possibly more than one control) and a *controlled qubit rotation*.

In general, whenever one has a controlled gate g with more than one control bit, one can simplify g by expressing its exponents $P_0(\beta) = \bar{n}(\beta)$ and $P_1(\beta) = n(\beta)$ in terms of $I_2(\beta)$ and $\sigma_z(\beta)$. Appendices C and D say more about this transformation. Appendix D discusses the transformation from the point of view of Linear Algebra. Appendix C uses the transformation to decompose a controlled gate with 2 controls into a SEO. Decompositions like the one in Appendix C were first discovered by the authors of Ref.[5]. However, their method of deriving such decompositions is very different from ours.

5. SEO-Length Optimizations

In previous sections, we have given an algorithm that can decompose an arbitrary unitary matrix into a SEO. But the SEO's generated by this algorithm may not be the shortest possible. In this chapter we will indicate several "SEO-length optimizations"—small adjustments to the algorithm which are either guaranteed or, at least, likely to produce shorter SEO's.

The group $U(N_S)$ of $N_S \times N_S$ unitary matrices has N_S^2 free (real) parameters. (It has N_S^2 complex entries for a total of $2N_S^2$ real parameters, but those parameters must satisfy N_S^2 orthonormality constraints.) In a CSD tree, level r has 2^{r-1} nodes so the number of nodes in the tree is $1 + 2 + 2^2 + \dots + 2^{N_B} = 2^{N_B+1} - 1 \approx 2N_S$. (Level $N_B + 1$ alone has N_S of the nodes, about half of them!) Each node yields $\text{Order}(N_S)$ elementary operations. Therefore, in general, a CSD tree yields a SEO whose length is $\text{Order}(N_S^2)$. In essence, each elementary operation of the SEO (except permutation operations like c-nots) carries (approximately) one of the N_S^2 free parameters.

But we know that certain families of matrices contained in $U(N_S)$ have internal symmetries which allow us to parametrize them with substantially fewer than N_S^2 parameters. And we know how to decompose some of these families (for example, the DFT matrices) into a SEO whose length is a polynomial in N_B . Can such SEOs be obtained with our CSD tree algorithm? As we shall see, the answer is yes, at least in some important cases like the DFT matrices. The trick is to introduce optimizations

into our algorithm which make the CSD tree degenerate into a simple string (or nearly one) of nodes. The other nodes never “grow”. For any node \mathcal{N} , if its right side matrices (i.e., $R(\vec{b}, \mathcal{N})$ in Fig.1, where $\vec{b} \in Bool^\lambda$ and λ is the level of \mathcal{N}) are all equal to the identity matrix, then \mathcal{N} bears no children on its right side. Ditto for its left side matrices. Note that in Fig.2, the first (ditto, second) quantum FFT expansion looks like what one would expect if a CSD tree were to degenerate into the string consisting of the rightmost (ditto, leftmost) node of each level of the tree. As we shall see, this is precisely what happens if we add the following optimizations to our CSD tree algorithm.

5(a) Lazy Ordering of Factors

Consider Eq.(4a.19). The operators $A_{\vec{b}}$ commute so this equation is valid regardless of the order in which the $A_{\vec{b}}$ are multiplied. Suppose that we multiply the $A_{\vec{b}}$ so that their subscripts \vec{b} are in a lazy ordering that starts with 00 on the right and ends with 01 on the left:

$$D = A_{01}A_{11}A_{10}A_{00} . \quad (5a.1)$$

After inserting Eqs.(4a.20) for the $A_{\vec{b}}$, this yields

$$D = \sigma_x(2)^{n(0)}\Gamma(\theta_{01})\sigma_x(2)^{n(1)}\Gamma(\theta_{11})\sigma_x(2)^{n(0)}\Gamma(\theta_{10})\sigma_x(2)^{n(1)}\Gamma(\theta_{00}) . \quad (5a.2)$$

We see that by ordering the $A_{\vec{b}}$ in this way, several c-nots cancel out. Only one c-not remains between adjacent Γ 's. There is no c-not to the right of the rightmost Γ in Eq.(5a.2), because we started the lazy ordering with $\vec{b} = 00$. There is only one c-not to the left of the leftmost Γ in Eq.(5a.2), because we ended the lazy ordering with $\vec{b} = 01$.

The above example assumes $N_B = 3$ and that we are decomposing a central matrix of type 1 (i.e., a central matrix which is a single D matrix). However, this method of shortening a SEO can also be used for the other two types of central matrices and for other values of N_B . Indeed, the other two types of central matrices are also decomposed by the algorithm into a product of $A_{\vec{b}}$ type matrices. Furthermore, it is possible to find for arbitrary N_B a lazy ordering of $Bool^{N_B}$ which starts with the zero vector $\vec{b} = 0$ and ends with a \vec{b} which has only one non-zero component.

5(b) Lightning Right Side Matrices

Suppose D is a 4×4 D matrix such that its two angles are equal. Hence,

$$D = \begin{bmatrix} cI_2 & sI_2 \\ -sI_2 & cI_2 \end{bmatrix} , \quad (5b.1)$$

where $c = \cos \theta$, $s = \sin \theta$ for some angle θ . Then for any 2×2 unitary matrix G_0 ,

$$(G_0 \oplus G_0)D(G_0^\dagger \oplus G_0^\dagger) = D . \quad (5b.2)$$

More generally, consider an $N \times N$ D matrix D . Let θ_a for $a \in Z_{0, \frac{N}{2}-1}$ be its angles, arranged in non-decreasing order (recall Appendix A). Suppose the first d_0 angles are equal to each other, the next d_1 angles are equal to each other but larger than the previous d_1 angles, and so on. Hence, $d_0 + d_1 + \dots + d_{M-1} = \frac{N}{2}$. We will call d_0, d_1, \dots, d_{M-1} the *degeneracies of the D matrix angles*. For $a \in Z_{0, M-1}$, let G_a be any $d_a \times d_a$ unitary matrix, and define G by

$$G = G_0 \oplus G_1 \oplus \dots \oplus G_{M-1} . \quad (5b.3)$$

Then

$$(G \oplus G)D(G^\dagger \oplus G^\dagger) = D . \quad (5b.4)$$

Suppose that

$$U = (L_0 \oplus L_1)D(R_0 \oplus R_1) \quad (5b.5)$$

is a CSD of a unitary matrix U . If G is a unitary matrix that satisfies Eq.(5b.4), define L'_j and R'_j for $j \in Bool$ by

$$L'_j = L_j G , \quad (5b.6)$$

$$R'_j = G^\dagger R_j . \quad (5b.7)$$

Then

$$U = (L'_0 \oplus L'_1)D(R'_0 \oplus R'_1) \quad (5b.8)$$

is also a CSD of U . Suppose D has M distinct angles with degeneracies d_0, d_1, \dots, d_{M-1} . Define $\Sigma_{-1} = 0$ and $\Sigma_a = d_0 + d_1 + \dots + d_a$ for all $a \in Z_{0, M-1}$. For $a \in Z_{0, M-1}$ and $j \in Bool$, suppose $R_j^{(a)}$ (ditto, $R_j^{(a)}$) are the d_a rows with row indices from Σ_{a-1} to $\Sigma_a - 1$ of R'_j (ditto, R_j). Then Eq.(5b.7) is equivalent to the equations

$$R_j^{(a)} = G_a R_j'^{(a)} , \quad (5b.9)$$

for $a \in Z_{0, M-1}$ and $j \in Bool$. For each $a \in Z_{0, M-1}$, we will choose the two matrices G_a and $R_0'^{(a)}$ on the right side of the last equation so that they are the Q and R matrices, respectively, in a QR decomposition of $R_0^{(a)}$. In a QR decomposition[16], the Q matrix is unitary so constructing G_a this way will make it unitary, as required by our original definition of it. For any rectangular matrix M , its principal diagonal are those entries of M for which the row and column indices are equal. In a QR decomposition, the R

matrix has zero entries below the principal diagonal so $R_0^{(a)}$ will have this property. In a QR decomposition, it is possible to multiply each row of the R matrix by a phase factor and the corresponding column of Q by the conjugate phase factor. Thus it is possible for each $a \in Z_{0,M-1}$ to do a QR decomposition in which all the terms along the principal diagonal of $R_0^{(a)}$ are non-negative. (Alternatively, one can make non-negative all the terms along the principal diagonal of R'_0 .)

By *lightening the right side matrices*, we will mean replacing the CSD Eq.(5b.5) by the CSD Eq.(5b.8), where G is chosen so that for all $a \in Z_{0,M-1}$, the $R_0^{(a)}$ have zero entries below the principal diagonal and they (or else R'_0) all have non-negative diagonal entries. The strategy of lightening the right side matrices is not guaranteed to make a CSD tree have fewer nodes. However, whenever it does achieve this goal, the reason why it does so is clear. The strategy tries to make zero as many entries of R'_0 as possible. This causes the CSD tree to grow mostly to the left. Its right side growth is stunted.

5(c) Extracting Phases From Complex D Matrices

For this optimization, it is convenient to generalize the definitions of a D matrix and of a central matrix. Henceforth, let a *D matrix* be any unitary matrix D which can be written in the form

$$D = \begin{bmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{bmatrix}, \quad (5c.1a)$$

where the $D_{i,j}$ for $i, j \in Bool$ are same-sized diagonal matrices. An equivalent definition: a D matrix is a matrix of the form

$$D = \sum_{\vec{a} \in Bool^{N_B-1}} U^{(\vec{a})} \otimes P_{\vec{a}}, \quad (5c.1b)$$

where the $U^{(\vec{a})}$ are unitary 2×2 matrices. What we previously called a D matrix will now be called a *real D matrix*. If we need to emphasize that a D matrix is of the new kind, we will describe it as a *complex D matrix*. Henceforth, let a *central matrix* be defined as before, except that the term “D matrix” in the previous definition of a central matrix should now be interpreted to mean a complex D matrix. What we previously called a central matrix will now be called a *bare central matrix* (we can’t call it a real central matrix because it could be a complex unitary diagonal matrix). If we need to emphasize that a central matrix is of the new kind, we will describe it as a *dressed central matrix*.

In previous sections, we showed that: (1) Given an arbitrary unitary matrix U , one can construct a CSD tree in which each node stores a bare central matrix such that U equals the product (in a particular order) of these central matrices. (2) Any bare central matrix can be decomposed into a SEO. In this section, we will show

(1) and (2), but with the words “bare central matrix” replaced by “dressed central matrix”.

To construct a CSD tree of dressed central matrices, one proceeds as follows. Suppose node \mathcal{N} at level $\lambda \in Z_{1, N_B-1}$ has children nodes \mathcal{N}_L and \mathcal{N}_R to its left and right, respectively. Node \mathcal{N} will store a dressed central matrix $Q(\mathcal{N})$, a collection of left side matrices $L(\vec{b}, \mathcal{N})$ for $\vec{b} \in Bool^\lambda$, and a collection of right side matrices $R(\vec{b}, \mathcal{N})$ for $\vec{b} \in Bool^\lambda$. One performs a normal CSD only on those side matrices $R(\vec{b}, \mathcal{N})$ and $L(\vec{b}, \mathcal{N})$ which are not complex D matrices. If a side matrix is a complex D matrix, one performs an aborted CSD. Given a subroutine $f(U, L_0, L_1, D, R_0, R_1)$ for doing CSD, with input U and outputs L_0, L_1, D, R_0, R_1 , an *aborted CSD* is defined as a pass through the subroutine that returns the identity matrix for the side matrices L_0, L_1, R_0, R_1 and it returns U for D .

So far we have shown (1)— how to construct a CSD tree of dressed central matrices. Next we will show (2)— how to decompose a dressed central matrix into a SEO. We will show how any dressed central matrix Q can be factored into

$$Q = \Delta_L Q' \Delta_R, \quad (5c.2)$$

where Q' is a bare central matrix with real entries and where Δ_L and Δ_R are diagonal unitary matrices (so they are also bare central matrices, though not necessarily real). In previous sections, we showed how to decompose any bare central matrix into a SEO. So we will be done proving (2) if we can show Eq.(5c.2)

Consider a 2×2 complex D matrix D . One can always express D as

$$D = \begin{bmatrix} ce^{i\Omega} & se^{i(\Omega+\omega_R)} \\ -se^{i(\Omega+\omega_L)} & ce^{i(\Omega+\omega_L+\omega_R)} \end{bmatrix}, \quad (5c.3)$$

where $\Omega, \omega_L, \omega_R, c, s$ are real numbers, and where $c^2 + s^2 = 1$. Eq.(5c.3) implies

$$D = \text{diag}(1, e^{i\omega_L}) \begin{bmatrix} c & s \\ -s & c \end{bmatrix} e^{i\Omega} \text{diag}(1, e^{i\omega_R}). \quad (5c.4)$$

Note that we can always choose $s \geq 0$. For if $s < 0$, then one can replace $s \rightarrow |s|$, $\omega_L \rightarrow \omega_L + \pi$ and $\omega_R \rightarrow \omega_R + \pi$. Likewise, we can always choose $c \geq 0$. For if $c < 0$, then one can replace $c \rightarrow |c|$, $\omega_L \rightarrow \omega_L + \pi$, $\omega_R \rightarrow \omega_R + \pi$ and $\Omega \rightarrow \Omega + \pi$. With $s \geq 0$ and $c \geq 0$, we can write $c = \cos \theta$ and $s = \sin \theta$ with $\theta \in [0, 90^\circ]$. It is a trivial exercise to solve for the parameters $\Omega, \omega_L, \omega_R$ and θ in terms of the entries of D .

Now consider an $N \times N$ complex D matrix D . Let

$$\Gamma_\sigma = \text{diag}(e^{i\omega_\sigma^{(0)}} e^{i\omega_\sigma^{(1)}}, \dots, e^{i\omega_\sigma^{(\frac{N}{2}-1)}}) \quad \text{for } \sigma \in \{L, R\}, \quad (5c.5)$$

$$\Gamma = \text{diag}(e^{i\Omega^{(0)}} e^{i\Omega^{(1)}}, \dots, e^{i\Omega^{(\frac{N}{2}-1)}}), \quad (5c.6)$$

$$C = \text{diag}(C_0, C_1, \dots, C_{\frac{N}{2}-1}), \quad (5c.7)$$

$$S = \text{diag}(S_0, S_1, \dots, S_{\frac{N}{2}-1}), \quad (5c.8)$$

where for any $j \in Z_{0, \frac{N}{2}-1}$ and any $\sigma \in \{L, R\}$, $\Omega^{(j)}$ and $\omega_\sigma^{(j)}$ are real, and $C_j = \cos \theta_j$, $S_j = \sin \theta_j$ with $\theta_j \in [0, 90^\circ]$. One can always express D as

$$D = [I_{\frac{N}{2}} \oplus \Gamma_L] \begin{bmatrix} C & S \\ -S & C \end{bmatrix} [\Gamma \oplus (\Gamma \Gamma_R)]. \quad (5c.9)$$

Note that this is really a special CSD. To harmonize with our similar CSD angle convention (see Appendix A), it is important that we choose the angles θ_j to be in the interval $[0, 90^\circ]$. As in the $N = 2$ case, it is trivial to solve for the parameters $\Omega^{(j)}$, $\omega_\sigma^{(j)}$ and θ_j in terms of the entries of D .

Now consider any dressed central matrix Q . We want to show that $Q = \Delta_L Q' \Delta_R$. This equation is trivially satisfied if Q is a diagonal unitary matrix so assume instead that Q is a direct sum of one or more complex D matrices. For example, suppose that

$$Q = D(00) \oplus D(01) \oplus D(10) \oplus D(11), \quad (5c.10)$$

where the matrices $D(\vec{b})$ for $\vec{b} \in \text{Bool}^2$ are complex D matrices. By Eq.(5c.9), for each $\vec{b} \in \text{Bool}^2$, one can write

$$D(\vec{b}) = \Delta_L(\vec{b}) D'(\vec{b}) \Delta_R(\vec{b}), \quad (5c.11)$$

where the matrices $\Delta_L(\vec{b})$ and $\Delta_R(\vec{b})$ are diagonal unitary matrices and $D'(\vec{b})$ is a real D matrix. $Q = \Delta_L Q' \Delta_R$ now follows if we set

$$\Delta_\sigma = \Delta_\sigma(00) \oplus \Delta_\sigma(01) \oplus \Delta_\sigma(10) \oplus \Delta_\sigma(11), \quad (5c.12)$$

for $\sigma \in \{L, R\}$ and

$$Q' = D'(00) \oplus D'(01) \oplus D'(10) \oplus D'(11). \quad (5c.13)$$

One can expand Δ_L , Q' and Δ_R into a SEO using the techniques of Section 4. Alternatively, one might do this only for Q' , and express Δ_L and Δ_R as a product of controlled phase factors. We end this section with a discussion of controlled phase factors and of their products.

Any $N_S \times N_S$ diagonal unitary matrix Δ can be expressed as a product of controlled phase factors. For example, suppose

$$\Delta = \text{diag}(e^{i\phi_{00}}, e^{i\phi_{01}}, e^{i\phi_{10}}, e^{i\phi_{11}}), \quad (5c.14)$$

where $\phi_{\vec{b}}$ is a real number for all $\vec{b} \in Bool^2$. Eq.(5c.14) can be rewritten as

$$\Delta = e^{i\phi_{00}\bar{n}(1)\bar{n}(0)} e^{i\phi_{01}\bar{n}(1)n(0)} e^{i\phi_{10}n(1)\bar{n}(0)} e^{i\phi_{11}n(1)n(0)} . \quad (5c.15)$$

By replacing all exponents $\bar{n}(\beta)$ with $1 - n(\beta)$, the last equation can be expressed in the form:

$$\Delta = e^{i\theta_{00}} e^{i\theta_{01}n(0)} e^{i\theta_{10}n(1)} e^{i\theta_{11}n(1)n(0)} , \quad (5c.16)$$

where the $\theta_{\vec{b}}$ are real numbers, linear combinations of the $\phi_{\vec{b}}$. The SEO of Eq.(5c.16) is preferable to that of Eq.(5c.15) because Eq.(5c.15) contains four 2-bit operations whereas Eq.(5c.16) contains only one. From the point of view of Linear Algebra, going from Eq.(5c.15) to Eq.(5c.16) is an example of changing from the $\mathcal{B}(n, \bar{n})$ to the $\mathcal{B}(n, 1)$ basis. See Appendix E for more information about this transformation.

In this paper, we use bare central matrices whose D matrices have angles contained in the interval $[0, 90^\circ]$. An interesting special case is when these angles are all either 0 or 90° exclusively. The entries of such central matrices are elements of $\{-1, 0, 1\}$. When the optimization being discussed in this section is turned ON, it is sometimes convenient to decompose such central matrices into a SEO by a method different from the one discussed in Section 4. See Appendix F for more information about this.

5(d)Permuting Bits before Each CSD

This optimization is only partially implemented in the current version of Qubiter, and it was not used to get any of the results of Section 6.

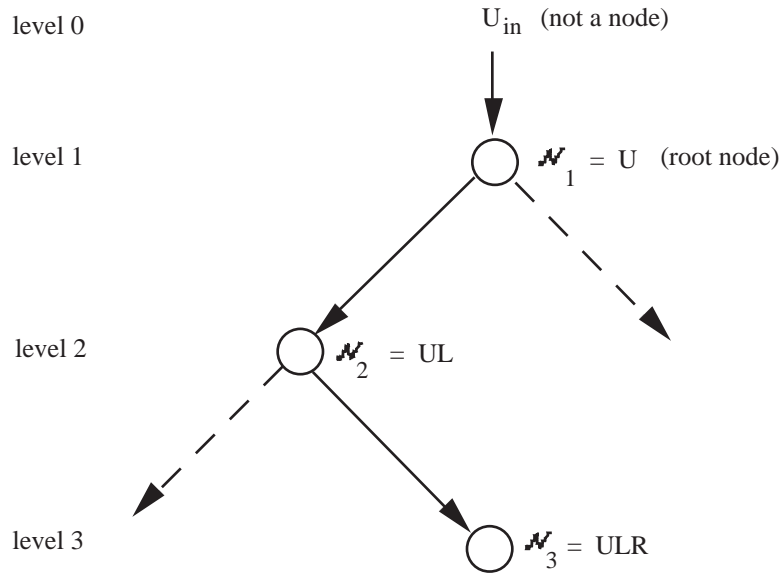


Fig.8 Subset of CSD tree.

We will start with an example and then generalize it. Suppose that $N_B \geq 3$ and that we are interested in finding the central matrix belonging to node $\mathcal{N}_3 = ULR$ of Fig.8. This being a binary tree, \mathcal{N}_3 has only one parent, call it $\mathcal{N}_2 = UL$. \mathcal{N}_2 in turn has just one parent $\mathcal{N}_1 = U$, which is the level 1 node (root node) of Fig.8. If U_{in} is the initial unitary matrix and $P(0, \mathcal{N}_1)$ is some permutation matrix that permutes bits (hence, $P(0, \mathcal{N}_1)$ is always expressible as a product of Exchangers), we do a CSD of $P(0, \mathcal{N}_1) \odot U_{in}$ instead of U_{in} :

$$P(0, \mathcal{N}_1) \odot U_{in} = [L(0, \mathcal{N}_1) \oplus L(1, \mathcal{N}_1)]D(0, \mathcal{N}_1)[R(0, \mathcal{N}_1) \oplus R(1, \mathcal{N}_1)] . \quad (5d.1a)$$

To get the central matrix belonging to \mathcal{N}_2 , we permute the bits of $L(0, \mathcal{N}_1)$ and $L(1, \mathcal{N}_1)$, and then do a CSD of the outcome:

$$P(a, \mathcal{N}_2) \odot L(a, \mathcal{N}_1) = [L(a0, \mathcal{N}_2) \oplus L(a1, \mathcal{N}_2)]D(a, \mathcal{N}_2)[R(a0, \mathcal{N}_2) \oplus R(a1, \mathcal{N}_2)] , \quad (5d.1b)$$

for all $a \in Bool$. Finally, to get the central matrix belonging to \mathcal{N}_3 , we permute the bits of $R(ab, \mathcal{N}_2)$ for all $a, b \in Bool$, and then do a CSD of the outcome:

$$P(ab, \mathcal{N}_3) \odot R(ab, \mathcal{N}_2) = [L(ab0, \mathcal{N}_3) \oplus L(ab1, \mathcal{N}_3)]D(ab, \mathcal{N}_3)[R(ab0, \mathcal{N}_3) \oplus R(ab1, \mathcal{N}_3)] , \quad (5d.1c)$$

for all $a, b \in Bool$. The central matrix $Q(\mathcal{N}_j)$ for $j \in Z_{1,3}$ is given by

$$Q(\mathcal{N}_1) = D(0, \mathcal{N}_1) , \quad (5d.2a)$$

$$Q(\mathcal{N}_2) = D(0, \mathcal{N}_2) \oplus D(1, \mathcal{N}_2) , \quad (5d.2b)$$

$$Q(\mathcal{N}_3) = D(00, \mathcal{N}_3) \oplus D(01, \mathcal{N}_3) \oplus D(10, \mathcal{N}_3) \oplus D(11, \mathcal{N}_3) . \quad (5d.2c)$$

One can define a permutation matrix $P(\mathcal{N}_j)$ for $j \in Z_{1,3}$ by

$$P(\mathcal{N}_1) = P(0, \mathcal{N}_1) , \quad (5d.3a)$$

$$P(\mathcal{N}_2) = P(0, \mathcal{N}_2) \oplus P(1, \mathcal{N}_2) , \quad (5d.3b)$$

$$P(\mathcal{N}_3) = P(00, \mathcal{N}_3) \oplus P(01, \mathcal{N}_3) \oplus P(10, \mathcal{N}_3) \oplus P(11, \mathcal{N}_3) . \quad (5d.3c)$$

It is useful to define matrices $Q'(\mathcal{N}_j)$ for $j \in Z_{1,3}$ by undoing the permutations that were performed to calculate $Q'(\mathcal{N}_j)$:

$$Q'(\mathcal{N}_1) = P^{-1}(\mathcal{N}_1) \odot Q(\mathcal{N}_1) , \quad (5d.4a)$$

$$Q'(\mathcal{N}_2) = [P(\mathcal{N}_2)P(\mathcal{N}_1)]^{-1} \odot Q(\mathcal{N}_2) , \quad (5d.4b)$$

$$Q'(\mathcal{N}_3) = [P(\mathcal{N}_3)P(\mathcal{N}_2)P(\mathcal{N}_1)]^{-1} \odot Q(\mathcal{N}_3) . \quad (5d.4c)$$

When the optimization presently being discussed is turned OFF, we associate with each node \mathcal{N} a central matrix $Q(\mathcal{N})$, and the product of these $Q(\mathcal{N})$ in some special order equals U_{in} . With this optimization turned ON, we associate with each \mathcal{N} a pair $(Q(\mathcal{N}), P(\mathcal{N}))$ consisting of a central matrix $Q(\mathcal{N})$ and a permutation matrix $P(\mathcal{N})$, and the product of the $Q'(\mathcal{N})$ equals U_{in} . Note that the matrices $Q'(\mathcal{N})$ are not necessarily central matrices.

The above discussion can be generalized to arbitrary N_B as follows. Assume \mathcal{N}_λ is a node at level $\lambda \in Z_{1, N_B}$. \mathcal{N}_λ has a string of predecessors $\mathcal{N}_{\lambda-1}, \mathcal{N}_{\lambda-2}, \dots, \mathcal{N}_1$, where \mathcal{N}_{j-1} is the single parent of \mathcal{N}_j and \mathcal{N}_1 is the tree's root node. Define $Bool^0 = \{0\}$. If $\lambda = 1$, let $U_{\lambda-1} = U_{in}$. If $\lambda > 1$, let $U_{\lambda-1}$ equal either $R(\vec{b}, \mathcal{N}_{\lambda-1})$ or $L(\vec{b}, \mathcal{N}_{\lambda-1})$ for some $\vec{b} \in Bool^{\lambda-1}$. For any $\vec{b} \in Bool^{\lambda-1}$, the D, R, L matrices of node \mathcal{N}_λ are obtained by applying to $U_{\lambda-1}$ a permutation matrix $P(\vec{b}, \mathcal{N}_\lambda)$ that permutes its bits and then doing a CSD on the outcome:

$$P(\vec{b}, \mathcal{N}_\lambda) \odot U_{\lambda-1} = [L(\vec{b}0, \mathcal{N}_\lambda) \oplus L(\vec{b}1, \mathcal{N}_\lambda)]D(\vec{b}, \mathcal{N}_\lambda)[R(\vec{b}0, \mathcal{N}_\lambda) \oplus R(\vec{b}1, \mathcal{N}_\lambda)] . \quad (5d.5)$$

Define the central matrix $Q(\mathcal{N}_\lambda)$ of node \mathcal{N}_λ by

$$Q(\mathcal{N}_\lambda) = \bigoplus_{\vec{b} \in Bool^{\lambda-1}} D(\vec{b}, \mathcal{N}_\lambda) . \quad (5d.6)$$

Define the permutation matrix $P(\mathcal{N}_\lambda)$ of node \mathcal{N}_λ by

$$P(\mathcal{N}_\lambda) = \bigoplus_{\vec{b} \in Bool^{\lambda-1}} P(\vec{b}, \mathcal{N}_\lambda) . \quad (5d.7)$$

Finally, define the matrix $Q'(\mathcal{N}_\lambda)$ of node \mathcal{N}_λ by

$$Q'(\mathcal{N}_\lambda) = [P(\mathcal{N}_\lambda) \dots P(\mathcal{N}_2)P(\mathcal{N}_1)]^{-1} \odot Q(\mathcal{N}_\lambda) . \quad (5d.8)$$

With this optimization ON, we associate a pair $(Q(\mathcal{N}), P(\mathcal{N}))$ with each node \mathcal{N} of the CSD tree, and the product of the (not necessarily central) matrices $Q'(\mathcal{N})$ in some order equals U_{in} .

How to best choose the permutation matrices $P(\mathcal{N})$ for this optimization remains an open question. One can carry out two main types of permutation searches: exhaustive or heuristic. An example of a partially exhaustive permutation search

(and the only permutation search implemented in the current version of Qubiter) is as follows. Try all possible bit permutations $P(\mathcal{N}_1)$ for the root node. Set $P(\mathcal{N})$ equal to the identity for all other nodes. Of all possibilities tried, find that which yields the shortest SEO. Obviously, one can try more complicated kinds of exhaustive searches. As for heuristic searches, for each node \mathcal{N} at level λ , one could try to find a permutation $P(\mathcal{N})$ which produces the smallest possible number of distinct CSD angles in the matrices $D(\vec{b}, \mathcal{N})$ for all $\vec{b} \in Bool^{\lambda-1}$. One expects that increasing the degeneracy in these CSD angles will make the “lightening right-side matrices” optimization more effective.

6. Qubiter

At present, Qubiter is a very rudimentary program. The current version—Qubiter1.1—is written in pure C++, and has no graphical user interface. All optimizations discussed in Section 5 except 5(d) have been fully implemented in Qubiter1.1. In its “compiling” mode, Qubiter takes as input a file with the entries of a unitary matrix and returns as output a file with a SEO. In its “decompiling” mode, it does the opposite: it takes a SEO file and returns the entries of a matrix. The lines in a SEO file are of 6 types.

(a) ROTY α ang

where $\alpha \in Z_{0, N_B-1}$ and ang is a real number. This signifies the rotation of qubit α about the Y axis by an angle of ang in degrees. In other words, $\exp(i\sigma_y(\alpha)ang\frac{\pi}{180})$. (Some people would call this a rotation by $2\ ang$ instead of ang).

(b) ROTZ α ang

This is the same as (a) except that the rotation is about the Z axis instead of the Y one.

(c) SIGX α

where $\alpha \in Z_{0, N_B-1}$. This signifies unconditional flipping (NOT) of qubit α . In other words, $\sigma_x(\alpha)$.

(d) CNOT α_1 $char_1$ α_2 $char_2$ \dots α_r $char_r$ β

where $r \geq 1$, $\alpha_1, \alpha_2, \dots, \alpha_r, \beta$ are distinct elements of Z_{0, N_B-1} and $char_1, char_2, \dots, char_r$ are elements of $\{T, F\}$. This signifies a controlled-not with r controls. First suppose $r = 1$. If $char_1$ is the character T , this signifies $\sigma_x(\beta)^{n(\alpha_1)}$. Read it as: “c-not: if α_1 is true, then flip β .” If $char_1$ is the character F , this signifies $\sigma_x(\beta)^{\overline{n(\alpha_1)}}$. Read it as: “c-not: if α_1 is false, then flip β .” Cases with $r > 1$ are defined analogously. For example, CNOT 0 T 1 F 2 signifies $\sigma_x(2)^{n(0)\overline{n(1)}}$. Read it as: “c-not: if bit 0 is true and bit 1 is false, then flip bit 2.”

- (e) PHAS $\quad ang$
 where ang is a real number. This signifies a phase factor $\exp(i(ang)\frac{\pi}{180})$. Thus, ang is an angle expressed in degrees.
- (f) CPHA $\quad \alpha_1 \quad char_1 \quad \alpha_2 \quad char_2 \quad \dots \quad \alpha_r \quad char_r \quad ang$
 where $r \geq 1$, $\alpha_1, \alpha_2, \dots, \alpha_r$ are distinct elements of Z_{0, N_B-1} , $char_1, char_2, \dots, char_r$ are elements of $\{T, F\}$, and ang is a real number. This signifies a controlled phase factor with r controls. For example, PHAS 0 T 1 F 90 would signify $[\exp(i90\frac{\pi}{180})]^{n^{(0)}\overline{n}^{(1)}}$. Read it as: “phase shift: if bit 0 is true and bit 1 is false, then shift phase by 90 degrees.” Thus, ang is an angle expressed in degrees.

CNOT with more than one control and CPHA with more than two are not elementary; i.e., they act on more than 2 bits. Strictly speaking, non-elementary operations should not be allowed in a SEO. However, Qubiter allows these 2 types because they are popular in the literature. We have indicated in previous sections how to decompose such controlled gates into SEOs.

Fig.9 shows the output of Qubiter for the 2, 3 and 4 bit Hadamard matrices, with all optimizations mention in Section 5 except 5(d) turned ON. Note that Qubiter is smart enough to realize that it is dealing here with a tensor product of 1 bit matrices. Fig.10 shows the output of Qubiter for the 2, 3 and 4 bit DFT matrices, with all optimizations mentioned in Section 5 except 5(d) turned ON. Note that the quantum FFT algorithm of Ref.[3] is exactly reproduced. Thus, the quantum FFT algorithm is a special case of our CSD tree algorithm. In both the Hadamard and DFT cases, the CSD tree degenerates into the string consisting of the leftmost node of each level of the tree.

In the future, we plan to introduce into Qubiter more optimizations and some quantum error correction. Much work remains to be done.

```

2 BITS:
ROTY 1 45.0000000
CPHA 1 T 180.000000
ROTY 0 45.0000000
CPHA 0 T 180.000000

```

```

3 BITS:
ROTY 2 45.0000000
ROTY 1 45.0000000
CPHA 1 T 180.000000
CPHA 2 T 180.000000
ROTY 0 45.0000000
CPHA 0 T 180.000000

```

```

4 BITS:
ROTY 3 45.0000000
ROTY 2 45.0000000
ROTY 1 45.0000000
CPHA 1 T 180.000000
CPHA 2 T 180.000000
CPHA 3 T 180.000000
ROTY 0 45.0000000
CPHA 0 T 180.000000

```

Fig.9 Output of Qubiter with input matrix equal to $\frac{1}{\sqrt{N_B}}H_{N_B}$ for $N_B = 2, 3, 4$. H_{N_B} is the N_B bit Hadamard matrix defined in Section 2(c).

```

2 BITS:
ROTY 1 45.0000000
CPHA 1 T 180.000000
CPHA 1 T 0 T 90.0000000
ROTY 0 45.0000000
CPHA 0 T 180.000000

```

```

3 BITS:
ROTY 2 45.0000000
CPHA 2 T 1 T 90.0000000
ROTY 1 45.0000000
CPHA 1 T 180.000000
CPHA 1 T 0 T 90.0000000
CPHA 2 T 180.000000
CPHA 2 T 0 T 45.0000000
ROTY 0 45.0000000
CPHA 0 T 180.000000

```

```

4 BITS:
ROTY 3 45.0000000
CPHA 3 T 2 T 90.0000000
ROTY 2 45.0000000
CPHA 2 T 1 T 90.0000000
CPHA 3 T 1 T 45.0000000
ROTY 1 45.0000000
CPHA 1 T 180.000000
CPHA 1 T 0 T 90.0000000
CPHA 2 T 180.000000
CPHA 2 T 0 T 45.0000000
CPHA 3 T 180.000000
CPHA 3 T 0 T 22.5000000
ROTY 0 45.0000000
CPHA 0 T 180.000000

```

Fig.10 Output of Qubiter with input matrix equal to $P_{BR}F_{N_B}$ for $N_B = 2, 3, 4$. F_{N_B} is the N_B bit DFT matrix defined in Section 2(d).

Appendix A. We Can Assume CSD Angles Are Non-decreasing Elements of $[0, 90^\circ]$

In this appendix we will show that: given any CSD of U as in Eq.(1b.1), one can always find another CSD of U for which the angles θ_i are in non-decreasing order and they are contained in the interval $[0, 90^\circ]$.

For simplicity, suppose the number of CSD angles is two. The discussion that follows can be easily generalized to any number of angles. For two CSD angles, Eq.(1b.1) becomes

$$U = \begin{bmatrix} L_0 & 0 \\ 0 & L_1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ 0 & c_2 & 0 & s_2 \\ -s_1 & 0 & c_1 & 0 \\ 0 & -s_2 & 0 & c_2 \end{bmatrix} \begin{bmatrix} R_0 & 0 \\ 0 & R_1 \end{bmatrix}. \quad (\text{A.1})$$

Suppose $c_2 < 0$. Then

$$U = \begin{bmatrix} L'_0 & 0 \\ 0 & L'_1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ 0 & |c_2| & 0 & -s_2 \\ -s_1 & 0 & c_1 & 0 \\ 0 & s_2 & 0 & |c_2| \end{bmatrix} \begin{bmatrix} R_0 & 0 \\ 0 & R_1 \end{bmatrix}, \quad (\text{A.2a})$$

where

$$L'_j = L_j \text{diag}(1, -1), \quad (\text{A.2b})$$

for $j = 0, 1$. Hence, we may assume $c_1, c_2 \geq 0$.

Suppose $c_1, c_2 \geq 0$ but $s_2 < 0$. Then

$$U = \begin{bmatrix} L_0 & 0 \\ 0 & L'_1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ 0 & c_2 & 0 & |s_2| \\ -s_1 & 0 & c_1 & 0 \\ 0 & -|s_2| & 0 & c_2 \end{bmatrix} \begin{bmatrix} R_0 & 0 \\ 0 & R'_1 \end{bmatrix}, \quad (\text{A.3a})$$

where

$$L'_1 = L_1 \text{diag}(1, -1), \quad (\text{A.3b})$$

$$R'_1 = \text{diag}(1, -1)R_1. \quad (\text{A.3c})$$

Hence we may assume $c_1, c_2, s_1, s_2 \geq 0$.

Finally, suppose that $\theta_1 > \theta_2$. Then

$$U = \begin{bmatrix} L'_0 & 0 \\ 0 & L'_1 \end{bmatrix} \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ 0 & c_1 & 0 & s_1 \\ -s_2 & 0 & c_2 & 0 \\ 0 & -s_1 & 0 & c_1 \end{bmatrix} \begin{bmatrix} R'_0 & 0 \\ 0 & R'_1 \end{bmatrix}, \quad (\text{A.4a})$$

where

$$L'_j = L_j \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (\text{A.4b})$$

$$R'_j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} R_j, \quad (\text{A.4c})$$

for $j = 0, 1$. Hence, we may assume that $\theta_1 \leq \theta_2$.

Appendix B. Proof of Quantum FFT Algorithm

Without pretending to do justice to Coppersmith's beautiful paper Ref.[3], here is a brief proof of his FFT algorithm, Eq.(2d.3), starting from the FFT algorithm that is usually found in textbooks. For simplicity, we will only consider the case $N_B = 4$. What follows can be easily generalized to arbitrary $N_B \geq 1$.

As usual, let

$$\omega = \exp(i\frac{2\pi}{N_S}) = \exp(i\frac{2\pi}{2^4}). \quad (\text{B.1})$$

Define matrices Ω, Ω^2, \dots by

$$\Omega = \text{diag}(1, \omega), \quad \Omega^2 = \text{diag}(1, \omega^2), \quad \dots \quad (\text{B.2})$$

Prior to Refs.[3], the FFT algorithm might have been stated like this: (see Refs.[2] and [19])

$$F_4 = \frac{1}{\sqrt{2}}(H \otimes I_8)[I_8 \oplus (\Omega^4 \otimes \Omega^2 \otimes \Omega^1)](I_2 \otimes F_3)P_4, \quad (\text{B.3})$$

$$F_3 = \frac{1}{\sqrt{2}}(H \otimes I_4)[I_4 \oplus (\Omega^4 \otimes \Omega^2)](I_2 \otimes F_2)P_3, \quad (\text{B.4})$$

$$F_2 = \frac{1}{\sqrt{2}}(H \otimes I_2)[I_2 \oplus \Omega^4](I_2 \otimes F_1)P_2, \quad (\text{B.5})$$

$$F_1 = \frac{1}{\sqrt{2}}H. \quad (\text{B.6})$$

The matrices P_4, P_3, P_2 are permutation matrices to be specified later. Although we could have written just a single equation that combined all four Eqs.(B.3) to (B.6), we have chosen not to do this, so as to make explicit the recursive nature of the beast. Note

$$\Omega^4 \otimes \Omega^2 \otimes \Omega = \text{diag}(1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7) = \omega^{2^2n(2)+2^1n(1)+2^0n(0)}. \quad (\text{B.7})$$

The last equation becomes clear when one realizes that $2^2n(2) + 2^1n(1) + 2^0n(0)$ operating on a state $|a_3, a_2, a_1, a_0\rangle$ gives the binary expansion of $d(0, a_2, a_1, a_0)$. Once Eq.(B.7) sinks into the old bean, it is only a short step to the realization that:

$$I_8 \oplus (\Omega^4 \otimes \Omega^2 \otimes \Omega) = \omega^{n(3)[2^2n(2)+2^1n(1)+2^0n(0)]} = e^{i[n(3)n(2)\phi_2+n(3)n(1)\phi_3+n(3)n(0)\phi_4]} , \quad (\text{B.8})$$

$$I_4 \oplus (\Omega^4 \otimes \Omega^2) = (\omega^2)^{n(2)[2^1n(1)+2^0n(0)]} = e^{i[n(2)n(1)\phi_2+n(2)n(0)\phi_3]} , \quad (\text{B.9})$$

$$I_2 \oplus (\Omega^4) = (\omega^4)^{n(1)[2^0n(0)]} = e^{i[n(1)n(0)\phi_2]} . \quad (\text{B.10})$$

The final step is to replace all tensor products that contain H by bit operators:

$$H \otimes I_8 = H(3) , \quad (\text{B.11})$$

$$I_2 \otimes H \otimes I_4 = H(2) , \quad (\text{B.12})$$

$$I_4 \otimes H \otimes I_2 = H(1) , \quad (\text{B.13})$$

$$I_8 \otimes H = H(0) . \quad (\text{B.14})$$

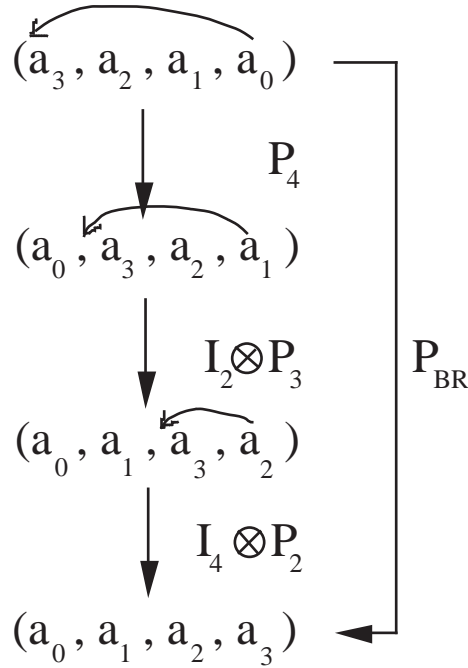


Fig.11 Permutation matrices P_4, P_3, P_2, P_{BR} that arise in the FFT algorithm for $N_B = 4$.

If all the permutation matrices of Eqs.(B.3) to (B.6) are combined into a single permutation P_{BR} , then

$$P_{BR} = (I_4 \otimes P_2)(I_2 \otimes P_3)P_4 . \quad (\text{B.15})$$

Let $\vec{a} \in \text{Bool}^4$ label the columns of a 16×16 matrix. As was known previous to Ref.[3], the matrices P_4, P_3, P_2 act as shown in Fig.11. Therefore, their product P_{BR} takes $\vec{a} = (a_3, a_2, a_1, a_0)$ to (a_0, a_1, a_2, a_3) ; i.e., P_{BR} reverses the bits of \vec{a} . P_{BR} is not an elementary operation. However, since it is a bit permutation, it can easily be expressed as a product of Exchangers, which are elementary.

Appendix C. Example of Decomposition of Controlled Gate into SEO

Suppose $N_B \geq 3$ and consider the following controlled gate:

$$g = B(2)^{n(1)n(0)} . \quad (\text{C.1})$$

Replacing n by $\frac{1}{2}[1 - \sigma_x]$, we get

$$g = B(2)^{\frac{1}{4}[1-\sigma_z(1)][1-\sigma_z(0)]} . \quad (\text{C.2})$$

Expanding the exponent of $B(2)$ in Eq.(C.2), and using identity Eq.(4a.9), we get

$$g = A_{00}A_{01}A_{11}A_{10} , \quad (\text{C.3})$$

where

$$A_{00} = B(2)^{\frac{1}{4}} , \quad (\text{C.4})$$

$$A_{01} = B(2)^{\frac{-1}{4}\sigma_z(0)} = B(2)^{\frac{1}{4}n(0)} B(2)^{\frac{-1}{4}} , \quad (\text{C.5})$$

$$A_{11} = B(2)^{\frac{1}{4}\sigma_z(1)\sigma_z(0)} = \sigma_x(1)^{n(0)} \odot B(2)^{\frac{1}{4}\sigma_z(1)} = [\sigma_x(1)^{n(0)} \odot B(2)^{\frac{-1}{4}n(1)}] B(2)^{\frac{1}{4}} , \quad (\text{C.6})$$

$$A_{10} = B(2)^{\frac{-1}{4}\sigma_z(1)} = B(2)^{\frac{1}{4}n(1)} B(2)^{\frac{-1}{4}} . \quad (\text{C.7})$$

The two $B(2)^{\frac{1}{4}}$ and two $B(2)^{\frac{-1}{4}}$ factors cancel each other out. We have ordered the factors $A_{\vec{b}}$ in Eq.(C.3) so that their subscripts $\vec{b} \in \text{Bool}^2$ are in a lazy ordering. (Ref.[5] does this too.) For the above case, lazy ordering is not important. But when g has more than two controls, lazy ordering permits some c-nots between adjacent $A_{\vec{b}}$'s to cancel. The same techniques can be used to decompose a controlled gate with more than two controls.

Appendix D. Transforming between $\mathcal{B}(n, \bar{n})$ and $\mathcal{B}(\sigma_z, I_2)$ bases

In this appendix, we will show how to transform between the $\mathcal{B}(n, \bar{n})$ and $\mathcal{B}(\sigma_z, I_2)$ bases of the set of $N_S \times N_S$ diagonal complex matrices.

Suppose that $N_B = 1$ and let

$$X = \begin{bmatrix} \bar{n}(0) \\ n(0) \end{bmatrix}, \quad (\text{D.1})$$

$$X' = \begin{bmatrix} 1 \\ \sigma_z(0) \end{bmatrix}. \quad (\text{D.2})$$

Then

$$X = M_1 X', \quad (\text{D.3})$$

where

$$M_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} H_1. \quad (\text{D.4})$$

Note that

$$M_1^{-1} = H_1. \quad (\text{D.5})$$

Next suppose that $N_B = 2$. Let

$$X = \begin{bmatrix} \bar{n}(1)\bar{n}(0) \\ \bar{n}(1)n(0) \\ n(1)\bar{n}(0) \\ n(1)n(0) \end{bmatrix} = \begin{bmatrix} \bar{n}(1) \\ n(1) \end{bmatrix} \otimes \begin{bmatrix} \bar{n}(0) \\ n(0) \end{bmatrix}, \quad (\text{D.6})$$

$$X' = \begin{bmatrix} 1 \\ \sigma_z(0) \\ \sigma_z(1) \\ \sigma_z(1)\sigma_z(0) \end{bmatrix} = \begin{bmatrix} 1 \\ \sigma_z(1) \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \sigma_z(0) \end{bmatrix}. \quad (\text{D.7})$$

Then

$$X = M_2 X', \quad (\text{D.8})$$

where

$$M_2 = \frac{1}{2^2} \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 1 & 1 & 1 \\ 01 & 1 & -1 & 1 & -1 \\ 10 & 1 & 1 & -1 & -1 \\ 11 & 1 & -1 & -1 & 1 \end{array} = \frac{1}{2^2} H_1 \otimes H_1 = \frac{1}{2^2} H_2 . \quad (\text{D.9})$$

Note that

$$M_2^{-1} = H_2 . \quad (\text{D.10})$$

The above results for $N_B = 1, 2$ can be easily generalized to any $N_B \geq 1$. One can prove using induction that for any N_B , the matrix M_{N_B} has entries given by

$$(M_{N_B})_{a,b} = \frac{1}{N_S} (-1)^{\vec{a} \cdot \vec{b}} , \quad (\text{D.11})$$

where $a, b \in Z_{0, N_S-1}$, $a = d(\vec{a})$ and $b = d(\vec{b})$. The matrix M_{N_B} is just the N_B bit Sylvester-Hadamard matrix divided by $1/N_S$.

If v and v' are N_S dimensional complex vectors such that

$$v^T X = v'^T X' , \quad (\text{D.12})$$

then

$$X = M_{N_B} X' , \quad (\text{D.13})$$

implies

$$v' = M_{N_B}^T v . \quad (\text{D.14})$$

Appendix E. Transforming between $\mathcal{B}(n, \bar{n})$ and $\mathcal{B}(n, I_2)$ bases

In this appendix, we will show how to transform between the $\mathcal{B}(n, \bar{n})$ and $\mathcal{B}(\sigma_z, I_2)$ bases of the set of $N_S \times N_S$ diagonal complex matrices.

Suppose that $N_B = 1$ and let

$$X = \begin{bmatrix} \bar{n}(0) \\ n(0) \end{bmatrix} , \quad (\text{E.1})$$

$$X' = \begin{bmatrix} 1 \\ n(0) \end{bmatrix} . \quad (\text{E.2})$$

Then

$$X = M_1 X' , \quad (\text{E.3})$$

where

$$M_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} . \quad (\text{E.4})$$

Note that

$$M_1^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} . \quad (\text{E.5})$$

Next suppose that $N_B = 2$. Let

$$X = \begin{bmatrix} \bar{n}(1)\bar{n}(0) \\ \bar{n}(1)n(0) \\ n(1)\bar{n}(0) \\ n(1)n(0) \end{bmatrix} = \begin{bmatrix} \bar{n}(1) \\ n(1) \end{bmatrix} \otimes \begin{bmatrix} \bar{n}(0) \\ n(0) \end{bmatrix} , \quad (\text{E.6})$$

$$X' = \begin{bmatrix} 1 \\ n(0) \\ n(1) \\ n(1)n(0) \end{bmatrix} = \begin{bmatrix} 1 \\ n(1) \end{bmatrix} \otimes \begin{bmatrix} 1 \\ n(0) \end{bmatrix} . \quad (\text{E.7})$$

Then

$$X = M_2 X' , \quad (\text{E.8})$$

where

$$M_2 = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & -1 & -1 & 1 \\ 01 & 0 & 1 & 0 & -1 \\ 10 & 0 & 0 & 1 & -1 \\ 11 & 0 & 0 & 0 & 1 \end{array} = M_1 \otimes M_1 . \quad (\text{E.9})$$

Note that

$$M_2^{-1} = M_1^{-1} \otimes M_1^{-1} . \quad (\text{E.10})$$

For any non-negative integers a, b , let $a\&b$ and $a \wedge b$ denote the same thing as they do in the C and C++ computer programming languages. That is, $a\&b$ is the result of bitwise and-ing of a and b , and $a \wedge b$ is the result of bitwise xor-ing of a and b (which is the same as bitwise mod(2) addition of a and b). For any $\vec{a} \in \text{Bool}^{N_B}$, if $a = d(\vec{a})$, we define $\|a\|_1 = \|\vec{a}\|_1 = \sum_{\mu=0}^{N_B-1} a_\mu$. In other words, $\|a\|_1$ is the number of ON bits in \vec{a} .

The above results for $N_B = 1, 2$ can be easily generalized to any $N_B \geq 1$. One can prove using induction that for any N_B , the matrix M_{N_B} has entries given by

$$(M_{N_B})_{a,b} = (-1)^{\|a \wedge b\|_1} \delta(a \& b, a), \quad (\text{E.11})$$

where $a, b \in Z_{0, N_B-1}$.

Appendix F. Central Matrices Whose Entries Are Elements of $\{-1, 0, 1\}$

In this paper, we use bare central matrices whose D matrices have angles contained in the interval $[0, 90^\circ]$. An interesting special case is when these angles are all either 0 or 90° exclusively. The entries of such central matrices are elements of $\{-1, 0, 1\}$.

First suppose that the central matrix Q is a single D matrix. Then we can express Q as

$$Q = \exp \left(i \sum_{\vec{b} \in \text{Bool}^{N_B-1}} \phi_{\vec{b}} \sigma_y \otimes P_{\vec{b}} \right), \quad (\text{F.1})$$

where the ϕ_b are all either 0 or 90° . Thus

$$Q = [e^{i\frac{\pi}{2}\sigma_y(N_B-1)}]_{\sum_{\vec{b} \in S} P_{\vec{b}}} = \prod_{\vec{b} \in S} [e^{i\frac{\pi}{2}\sigma_y(N_B-1)}]_{P_{\vec{b}}}, \quad (\text{F.2a})$$

where

$$S = \{\vec{b} | \vec{b} \in \text{Bool}^{N_B-1}, \phi_{\vec{b}} = \frac{\pi}{2}\}. \quad (\text{F.2b})$$

Hence, Q can always be expressed as a product of controlled qubit rotations. If $X = \sum_{\vec{b} \in S} P_{\vec{b}}$, then Q always acts on more than 2 bits at a time except when (1) $X = 1$ or when (2) $X = P_b(\alpha)$ for some $b \in \text{Bool}$ and some $\alpha \in Z_{0, N_B-1}$ such that $\alpha \neq N_B-1$. In the first case, Q is just a simple qubit rotation. In the second case

$$Q = [e^{i\frac{\pi}{2}\sigma_y(N_B-1)}]_{P_b(\alpha)}. \quad (\text{F.3})$$

Next suppose that Q is a direct sum of several D matrices. As explained in Section 4(b), such a Q can be obtained by applying a bit permutation matrix to a Q that, like the one in Eq.(F.1), is a single D matrix.

Thus, when Q is a direct sum of one or more D matrices, Q always acts on more than 2 bits except when it is a simple qubit rotation or when it has the form

$$Q = [e^{i\frac{\pi}{2}\sigma_y(\gamma)}]_{P_b(\alpha)}, \quad (\text{F.4})$$

where α and γ are distinct elements of Z_{0,N_B-1} and $b \in Bool$. Below, we will show that Eq.(F.4) implies

$$Q = \sigma_x(\gamma)^{P_b(\alpha)}(-1)^{\bar{n}(\gamma)P_b(\alpha)} = (-1)^{n(\gamma)P_b(\alpha)}\sigma_x(\gamma)^{P_b(\alpha)} . \quad (\text{F.5})$$

Hence, such a Q can be expressed as a product of a c-not and a controlled phase shift with 2 controls. Whenever the optimization of Section 5(c) (Extracting Phases from Complex D Matrices) is ON, we will decompose any Q of the form Eq.(F.4) into the SEO of Eq.(F.5).

Note that

$$e^{i\frac{\pi}{2}\sigma_y} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \quad (\text{F.6})$$

$$(-1)^{\bar{n}} = \text{diag}(-1, 1) , \quad (\text{F.7})$$

$$(-1)^n = \text{diag}(1, -1) . \quad (\text{F.8})$$

Thus,

$$e^{i\frac{\pi}{2}\sigma_y} = \sigma_x(-1)^{\bar{n}} = (-1)^n\sigma_x . \quad (\text{F.9})$$

(n changes to \bar{n} because $\sigma_x\sigma_z = -\sigma_z\sigma_x$.) Eq.(F.5) follows directly from Eq.(F.9).

Note that we could also decompose the Q of Eq.(F.4) using the techniques of Section 4 and Appendix C:

$$\begin{aligned} e^{i\frac{\pi}{2}\sigma_y(\gamma)n(\alpha)} &= e^{i\frac{\pi}{4}\sigma_y(\gamma)}e^{-i\frac{\pi}{4}\sigma_y(\gamma)\sigma_z(\alpha)} \\ &= [e^{i\frac{\pi}{4}\sigma_y(\gamma)}][\sigma_x(\gamma)^{n(\alpha)} \odot e^{-i\frac{\pi}{4}\sigma_y(\gamma)}] . \end{aligned} \quad (\text{F.10})$$

Neither decomposition Eq.(F.5) nor decomposition Eq.(F.10) uses controlled qubit rotations. The SEO of Eq.(F.5) is shorter than the SEO of Eq.(F.10). Eq.(F.5) uses a controlled phase shift whereas Eq.(F.10) doesn't. Since controlled phase shifts are common when the optimization of Section 5(c) is ON, it is natural to use Eq.(F.5) when said optimization is ON.

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FIGURE CAPTIONS:

FIG.1 A CSD binary tree.

FIG.2 Pictorial representation of quantum FFT algorithm. For simplicity, we only show the Hadamard matrices. White (unshaded) regions inside a matrix represent zero entries.

FIG.3 Circuit symbols for the 4 different types of c-nots.

FIG.4 Four equivalent circuit diagrams for Exchanger.

FIG.5 Four equivalent circuit diagrams for Twin-to-twin-er.

FIG.6 Circuit symbol for Exchanger.

FIG.7 Circuit diagram for Eq.(3b.7).

FIG.8 Subset of a CSD tree.

FIG.9 Output of Qubiter with input matrix equal to $\frac{1}{\sqrt{N_B}}H_{N_B}$ for $N_B = 2, 3, 4$. H_{N_B} is the N_B bit Hadamard matrix defined in Section 2(c).

FIG.10 Output of Qubiter with input matrix equal to $P_{BR}F_{N_B}$ for $N_B = 2, 3, 4$. F_{N_B} is the N_B bit DFT matrix defined in Section 2(d).

FIG.11 Permutation matrices P_4, P_3, P_2, P_{BR} that arise in the FFT algorithm for $N_B = 4$.