Problem Set - 5

Autumn 2019

MATHEMATICS-I(MA10001) //2019

1. Find $\dfrac{dz}{dt}$ for the following functions,

   (a) $z = f(x, y) = x^2 + xy$, where $x(t) = e^t$, $y(t) = \sin(t)$.

   (b) $z = f(x, y) = xy$, where $x(t) = \cos(t)$, $y(t) = \sin(t)$.

2. If $z = x^2 + y^2$, and $x(u, v) = u\cos(v)$, $y(u, v) = u\sin(v)$, find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

3. (a) Using implicit differentiation, find $\dfrac{dy}{dx}$ from the followings:

   i. $y^x = x^y$,

   ii. $\sin(xy) - e^{xy} - x^2y = 0$

   iii. $xe^y + ye^x - e^{xy} = 0$

   iv. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

   (b) Using implicit differentiation, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ from the followings:

   i. $xy^2z^2 + \sin(yz) - e^{xz^2} = c$,

   ii. $x\tan^{-1}(\frac{y}{z}) + y\tan^{-1}(\frac{z}{x}) + z\tan^{-1}(\frac{x}{y}) = c$

4. Check whether the following functions are homogeneous or not, if so, determine the degree of the function:

   (a) $\tan^{-1}\dfrac{y}{x} + \sin^{-1}\dfrac{x}{y}$

   (b) $x^{2/3}y^{4/3}\tan\dfrac{y}{x}$

   (c) $\dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

   (d) $x^2y^2 + xy^3 + x^2y + x^3y$

   (e) $\dfrac{\sqrt{x^6 + y^6}}{x + y}$

5. If $w = f(\dfrac{y - x}{xy}, \dfrac{z - y}{zy})$, then show that $x^2\dfrac{\partial w}{\partial x} + y^2\dfrac{\partial w}{\partial y} + z^2\dfrac{\partial w}{\partial z} = 0$.

6. Show that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 3f$, where $f(x, y) = x^3 + 3x^2y + xy^2 + 9y^3$.

7. If $u = \sin^{-1}(\dfrac{x^3 + y^3}{x - y})$, then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\tan u$.

8. Let $f(x,y)$ be a homogeneous function of x and y of degree n having continuous first order partial derivatives and $u = (x^2+y^2)^{-n/2}$, then show that $x\dfrac{\partial}{\partial x}(fu) + y\dfrac{\partial}{\partial y}(fu) = 0$.

9. If $u = \frac{1}{y}[\phi(ax+y)+\phi(ax-y)]$, then show that $\frac{\partial^2 u}{\partial x^2} = \frac{a^2}{y^2}[\frac{\partial}{\partial y}(y^2\frac{\partial u}{\partial y})]$.

10. If $u = ze^{ax+by}$, where z is a homogeneous function in x and y of degree n, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = (ax+by+n)u$.

11. If $u = \tan^{-1}(\dfrac{x^3+y^3}{x-y})$, then show that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = (1 - 4\sin^2 u)\sin 2u$.

12. If $u = \sin^{-1}\sqrt{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$, then show that

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right).$$

13. If $u = \dfrac{(ax^3+by^3)^n}{3n(3n-1)} + xf(\dfrac{y}{x})$, then prove that
$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = (ax^3+by^3)^n$.

14. If $z = x^m f(\dfrac{y}{x}) + y^n g(\dfrac{x}{y})$, then show that
$x^2z_{xx} + 2xyz_{xy} + y^2z_{yy} + mnz = (m+n-1)(x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y})$.

15. If $u = x\phi(x+y) + y\psi(x+y)$, then show that $u_{xx} - 2u_{xy} + u_{yy} = 0$.

16. If $u = x\phi(\dfrac{y}{x}) + \psi(\dfrac{y}{x})$, then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = x\phi(\dfrac{y}{x})$ and $(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})^2u = 0$.

17. If $z(x,y) = \dfrac{(x^2+y^2)^n}{2n(2n-1)} + x\phi(\dfrac{y}{x}) + \psi(\dfrac{y}{x})$, then using Euler's theorem, show that $x^2z_{xx} + 2xyz_{xy} + y^2z_{yy} = (x^2+y^2)^n$.

18. Let $u(x, y)$ be such that all its second order partial derivatives exists. If $x = r\cos\theta, y = r\sin\theta$, then show that

$$r^2\frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial \theta^2} - r\frac{\partial u}{\partial r} = (x^2 - y^2)(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}) + 4xy\frac{\partial^2 u}{\partial x \partial y}.$$

19. If z be a differentiable function of x and y (rectangular cartesian co-ordinates) and let $x = r\cos\theta, y = r\sin\theta (r, \theta$ are polar co-ordinates), then show that

    (a) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial z}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial \theta^2}.$

    (b) $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = \left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2.$