Tutorial Sheet - 4

Autumn 2019

MATHEMATICS-I(MA10001)

1. Show that the functions

(a) $$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{|x|+|y|}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

(b) $$f(x,y) = \begin{cases} x \sin \dfrac{1}{x} + y \sin \dfrac{1}{y}, & \text{if } x \neq 0, y \neq 0 \\ x \sin \dfrac{1}{x}, & \text{if } x \neq 0, y = 0 \\ y \sin \dfrac{1}{y}, & \text{if } x = 0, y \neq 0 \\ 0, & \text{if } x = 0, y = 0 \end{cases}$$

are continuous at $(0,0)$, but $f_x(0,0)$ and $f_y(0,0)$ do not exist.

2. Show that the following functions

(a) $$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

(b) $$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

possesses partial derivatives at $(0,0)$, though it is not continuous at $(0,0)$.

3. Find $f_x(x,y)$ and $f_y(x,y)$ for the followings :

(a) $$f(x,y) = x^2 \tan^{-1}(\frac{y}{x}) - y^2 \tan^{-1}(\frac{x}{y})$$

(b) $$f(x,y) = \frac{\sin y + \cos x}{x^3+y^3}$$

(c) $f(x,y) = \dfrac{e^{(x^2+y^2)}}{\sqrt{x^2+y^2}} - \log\dfrac{x}{y}$

(d) $f(x,y) = \dfrac{3x^2+2xy+5y^2}{x^2+xy}$

(e) $f(x,y) = \sinh\left(\dfrac{xy}{x^2+y^2}\right)$

(f) $f(x,y) = \dfrac{x^3+3y^3}{x^2-4y^2}$

4. Find $f_x(0,0)$, $f_y(0,0)$, $f_x(0,y)$ and $f_y(x,0)$ for the followings :

(a) $f(x,y) = \begin{cases} \dfrac{x^3y}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x^2+y^2 = 0 \end{cases}$

(b) $f(x,y) = \log(1+xy)$,

(c) $f(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 0 \text{ or } \textit{both } x = 0 \textit{ and } y = 0 \\ 0, & \text{Otherwise} \end{cases}$

(d) $f(x,y) = e^{x-y} - e^{y-x}$

5. Discuss the differentiability of the following functions at $(0,0)$.

(a) $f(x,y) = \begin{cases} x^2\sin\dfrac{1}{x} + y^2\sin\dfrac{1}{y}, & \text{if } x \neq 0, y \neq 0 \\ x^2\sin\dfrac{1}{x}, & \text{if } x \neq 0, y = 0 \\ y^2\sin\dfrac{1}{y}, & \text{if } x = 0, y \neq 0 \\ 0, & \text{if } x = 0, y = 0 \end{cases}$

(b) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

(c) $f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

(d) $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

6. Show that the following functions

(a) $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$

(b) $f(x,y) = (xy)^{\frac{1}{3}}$

are continuous, possess first order partial derivatives but are not differentiable at the origin.

7. Prove that the function $f(x,y) = \sqrt{|xy|}$ is not differentiable at $(0,0)$, but that f_x and f_y both exists at origin and have the value 0. Show that f_x and f_y are not continuous at the origin.

8. For the functions

(a) $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

(b) $f(x,y) = \begin{cases} (x^2+y^2)\tan^{-1}\dfrac{y}{x}, & \text{if } x \neq 0 \\ \frac{\pi}{2}y^2, & \text{if } x = 0 \end{cases}$

(c) $f(x,y) = \begin{cases} y\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

Check that $f_{xy}(0,0) \neq f_{yx}(0,0)$. Also check the differentiability of the function $f(x,y)$ at the origin.

9. Find the total differential of the following functions.

(a) $w = \dfrac{2z^3}{3(x^2+y^2)}$

(b) $z = \tan^{-1}\dfrac{x}{y}$

(c) $u = \dfrac{e^{\tan^{-1}(3x+4y+5z)}}{1+\tan(3x+4y+5z)}$

(d) $w = \dfrac{\cos(xyz)}{y^2\ln(x^2z)+x^3y^3}$

(e) $w = \dfrac{x^3\sin y + y^3\cos x}{e^x\ln y + \sin y\ln x}$

(f) $u = \dfrac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}$

(g) $w = \dfrac{x\ln(xy)-\sin(y+2z)}{x^2+y^2+z^2}$

(h) $w = e^{\frac{x}{y}} + e^{\frac{z}{y}}$

(i) $z = \dfrac{xe^{\cos(xy)}}{e^{(x^2+y^2)}}$

(j) $z = \log(\sin\sqrt{y^2-\dfrac{x^2}{2}})$