

1. Determine the following limits using L'Hospital rule:

a) $\lim_{x \rightarrow \infty} (1 - \frac{1}{2x})^{x+1}$

b) $\lim_{x \rightarrow 0^+} x^{\sin x}$

c) $\lim_{x \rightarrow 0} (\frac{\sin x}{x})^{\frac{1}{x}}$

d) $\lim_{x \rightarrow 0} (\frac{1}{2-2\cos x} - \frac{1}{x^2})$

e) $\lim_{x \rightarrow 0} |\sin x|^x$

f) $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\log(\sin x)}{(\pi-x)^2}$

g) $\lim_{x \rightarrow 0} (\frac{2^x+3^x}{2})^{\frac{1}{x}}$

h) $\lim_{x \rightarrow \infty} \{x - \sqrt[n]{(x-1)(x-2)\dots(x-n)}\}$

2. a) Find the value of a for which the limit

$$\lim_{x \rightarrow 0} \frac{\sin(ax) - \sin x - x}{x^3}$$

is finite and evaluate the limit.

b) Find the values of a and b such that

$$\lim_{x \rightarrow 0} \frac{\cos(ax) - b}{2x^2} = -1.$$

3. Use Taylor's theorem to prove that

a) $x - \frac{x^2}{2} < \log(1+x) < x$ for $x > 0$.

b) $\cos x \geq 1 - \frac{x^2}{2}$ for $-\pi < x < \pi$.

c) $1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$ for $x > 0$.

4. Let $c \in \mathbb{R}$ and a real function f be such that f'' is continuous on some neighbourhood of c . Prove that

$$\lim_{h \rightarrow 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

5. Let $a \in \mathbb{R}$ and a real function f defined on some neighbourhood $N(a)$ of a such that f'' is continuous at a and $f''(a) \neq 0$. Prove that $\lim_{h \rightarrow 0} \theta = \frac{1}{2}$, where θ is given by $f(a+h) = f(a) + hf'(a+\theta h)$ ($0 < \theta < 1$).

6. Each of the series in the following is the value of the Taylor series at $x = 0$ of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

a) $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \dots$

b) $\frac{2}{3} - \frac{4}{18} + \frac{8}{81} - \dots$

c) $\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \dots$

7. Using Taylor series expansion, evaluate

a) $\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$

b) $\lim_{x \rightarrow 0} \frac{xe^x - \log(1+x)}{x^2}$

c) $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^2 \tan x}$

d) $\frac{\cosh x - \cos x}{x \sin x}$

8. If f is continuous at x_0 , and there are constants a_0 and a_1 such that

$$\lim_{x \rightarrow x_0} \frac{f(x) - a_0 - a_1(x - x_0)}{x - x_0},$$

then prove that $a_0 = f(x_0)$, f is differentiable at x_0 , and $f'(x_0) = a_1$.

9. Find the Maclaurin's infinite series expansion for

a) $e^x, x \in \mathbb{R}$

b) $\log(1+x), x \in (-1, 1]$.

c) $e^x \cos x, x \in \mathbb{R}$

10. Obtain the fourth degree Taylor's polynomial approximation to $f(x) = e^{2x}$ about $x = 0$. Find the maximum error when $0 \leq x \leq 0.5$.

11. Using Taylor's series find the approximate value of a) $\sqrt{1.5}$ and b) $\cos 31^\circ$.

12. Obtain the Maclaurin's series expansion of $f(x) = \sin(m \sin^{-1} x)$, where m is a constant.

13. For the Taylor's polynomial approximation of degree less than or equal to n about the point $x = 0$ for the function e^x , determine the value of n such that the error satisfies $|R_n(x)| \leq 0.005$, when $-1 \leq x \leq 1$.

14. Can the function $f(x)$ defined by $f(x) = \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$ be expanded by Maclaurin's theorem?

15. Find the Taylor's series expansion of $\sin^2 x$ upto five terms with Lagrange's form of remainder.

16. Find the Maclaurin's series expansion of $\tan^{-1} x$ upto four terms with Lagrange's form of remainder.