Lec-7.

Total differential.

$y = f(x)$ $\qquad$ $z = f(x, y)$

$dy = f'(x)\,dx$ $\qquad$ $dz = f_x(x,y)\,dx + f_y(x,y)\,dy$

$\Downarrow$ when 'dx' is smally it measure the changes in the functional values of 'f'.

$\Downarrow$ when dx, dy are small it capture the change in the functional values.

Exp. Find total differential of $f(x,y) = x^2 + 3xy - y^2$.

If x changes from 2 to 2.05, y changes from 3 to 2.96, compare $\boxed{\Delta z}$, and $\boxed{dz}$.

$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$

Ans: $dz = 0.65$, $\Delta z = 0.6449$

Meaning of differentiability for 1-variable function.

$$\frac{\Delta y}{\Delta x} - f'(x) = \varepsilon$$

since $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$

$$\boxed{\Delta y = \Delta x f'(x) + \varepsilon \Delta x \quad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0}$$

So $z = f(x,y)$ is differentiable if $\exists\ \varepsilon_1, \varepsilon_2$ s.t.

$$\Delta z = \underbrace{f_x(x,y)\Delta x + f_y(x,y)\Delta y} + \underbrace{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$$

'error term'

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$

$\varepsilon_2 \to 0$ as $\Delta y \to 0$

$$\boxed{\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$$

Equivalent.

$$\frac{\Delta z - dz}{\sqrt{\Delta x^2 + \Delta y^2}} \to 0 \quad \text{as } \Delta x \to 0, \ \Delta y \to 0$$

$$\Vert$$

$$\frac{\Delta z - dz}{\sqrt{\Delta x^2 + \Delta y^2}} = \frac{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$= \varepsilon_1 \frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} + \varepsilon_2 \frac{\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}$$

To test / determine the differentiability of a fc $f(x,y)$ we have the following condition:

Thm. If f_x and f_y exist near (a,b) and continuous at (a,b) then f is differentiable at (a,b).

Exp. $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$.

Find the tangent plane at $(1,0)$, total differential and use it to approximate

$f(1.1, -0.1)$.

$L(x,y) = x + y$

$f(1.1, -0.1) \approx 1$

$\parallel$

$1.1 \times e^{-0.11} \approx 0.98542$

For 1-variable Chain rule.

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$$

$f(x, y)$

For each time 't'
your position is
$(x(t), y(t))$ on
the google map.

$$z = f(x(t), y(t))$$ which
measure
the height
from sea
level.

Q. How must z is changing
with time 't'.

$\cancel{g(t)}$ $$z(t) = f(x(t), y(t))$$

$$z'(t) = \lim_{h \to 0} \frac{z(t+h) - z(t)}{h}$$

$$= \lim_{h \to 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t))}{h}$$

$$= \lim_{h \to 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t+h))}{h} + \lim_{h \to 0} \frac{f(x(t), y(t+h)) - f(x(t), y(t))}{h}$$

$$= \, ?$$

Thm. Suppose $z = f(x,y)$ is a diff. f: of x and y where $x = g(t)$, $y = h(t)$ are both diff. fns of 't'. Then z is diff. fn. of 't' and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Pf. Since z is diff.

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

and $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ when $\Delta x \to 0$, $\Delta y \to 0$

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

If $\Delta t \to 0$ $\boxed{\text{then } \varepsilon_1 \to 0}$

then $\Delta x = g(t+\Delta t) - g(t) \to 0$ since g is diff.

Illy $\Delta y = h(t+\Delta t) - h(t) \to 0$ since h is dif.

$$\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

$$+ \left(\lim_{\Delta t \to 0} \varepsilon_1\right)\left(\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\right)$$

$$+ \left(\lim_{\Delta t \to 0} \varepsilon_2\right)\left(\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}\right)$$

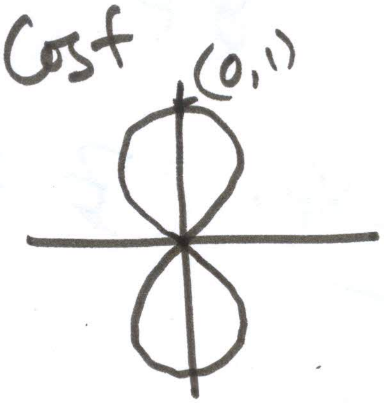
$$\Rightarrow \frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

$$+ 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt}$$

Exp. $z = x^2 y + 3xy^4$

$x = \sin 2t, \; y = \cos t$

Find $\frac{dz}{dt}$ at $t = 0$.



Ans. Ans. 6.

Verify.

Consider $z = f(x, y)$

where $x = g(s, t)$

$y = h(s, t)$

Q. $\frac{\partial z}{\partial s} = ?$, $\frac{\partial z}{\partial t} = ?$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

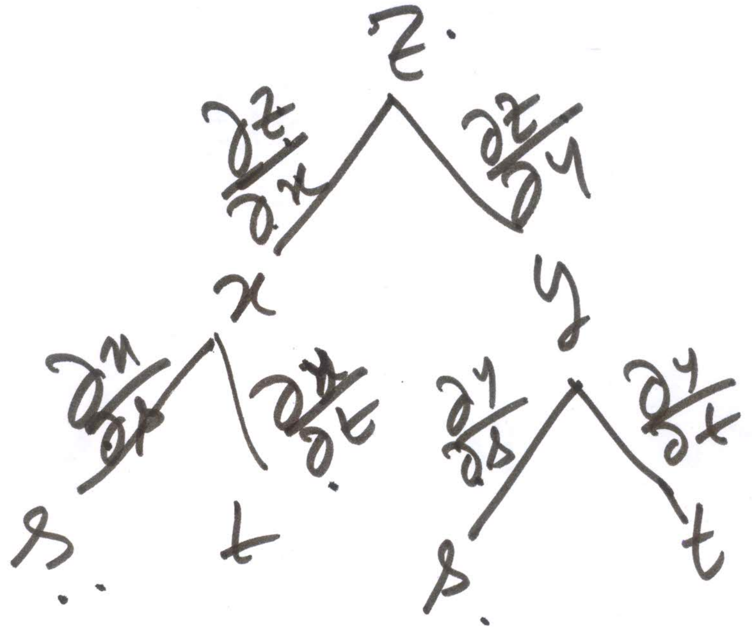
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Exp. $z = e^x \sin y$

where $x = st^2$

$y = s^2 t$

Compute $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$.



Tree representation.

Let $z = f(x, y, u, v)$

s.t. $y = f(x)$, $u = g(x)$, $v = h(x)$

$$\frac{\partial z}{\partial x} = ??$$