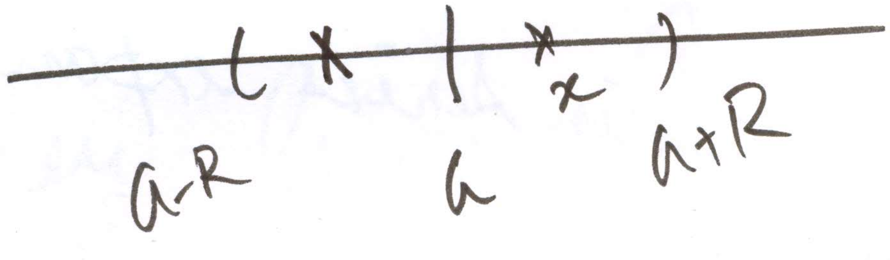
Lecture ~~3~~ 4

Approximating a 'well-behaved' function f by the Taylor series

$$f(x) = T_n(x) + R_n(x),$$

$$|x-a| < R$$



$$= \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{T_n(x)} + \underbrace{\frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)}_{= R_n(x)},$$

$$a < \xi < x$$

Maclaurin's Series.

Set $a = 0$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \boxed{\frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x)}$$

$$0 < \theta < 1$$

Exp. Maclaurin Series expansion of $e^x = f(x)$.

Then $f^{(n)}(0) = 1 \quad \forall n$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots\cdots$$

Meaning of Convergence of a series.

$$a_1 + a_2 + a_3 + \cdots\cdots = \sum_{n=1}^{\infty} a_n, \quad a_n \in \mathbb{R}$$

Ratio Test — Chalk Board

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

Set $x = 1$

$$a_n = \frac{1}{n!}$$

$$a_{n+1} = \frac{1}{(n+1)!}$$

Then $$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{n+1}$$

$$\to \underset{\underset{L}{\parallel}}{0} \text{ as } n \to \infty$$

$\therefore$ the Radius of Convergence $= \infty$

Exp.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots, \quad -1 < x < 1$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n} = 1 + 2x + 3x^2 + 4x^3 + \cdots, \quad -1 < x < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x \leq 1$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad -1 \leq x \leq 1.$$

L'Hospital Rule:

discovered Swiss John Bernoulli 1667 - 1748

named after French Marquis de L'Hospital 1661 - 1704.

Suppose f & g are diff. fns and $g'(x) \neq 0$ on an open interval I that contain 'c' (except possibly at 'c').

Let $\lim_{x \to c} f(x) = 0$ & $\lim_{x \to c} g(x) = 0$

or $\lim_{x \to c} f(x) = \pm\infty$ & $\lim_{x \to c} g(x) = \pm\infty$

Then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

Indeterminate Form

$$f(x) = \frac{\ln x}{x-1}$$

$$\lim_{x \to 1} f(x) = ?$$

$$\lim_{x \to 0} \frac{\sin x}{x} = ?$$

$$\lim_{x \to \infty} \frac{x^2-1}{2x^2+1} = \frac{1}{2}$$

Indeterminate product.

$$\lim_{x \to c} f(x) = 0 \ \& \ \lim_{x \to c} g(x) = \infty$$

$$\lim_{x \to c} f(x) g(x) = ?$$

$$\| $$

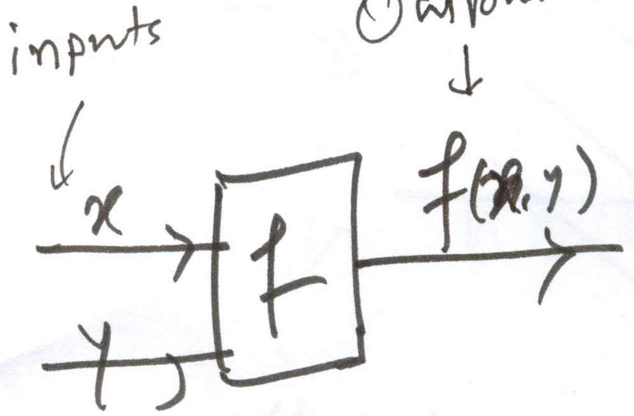
$$\frac{f(x)}{1/g(x)} = \frac{g(x)}{1/f(x)}$$

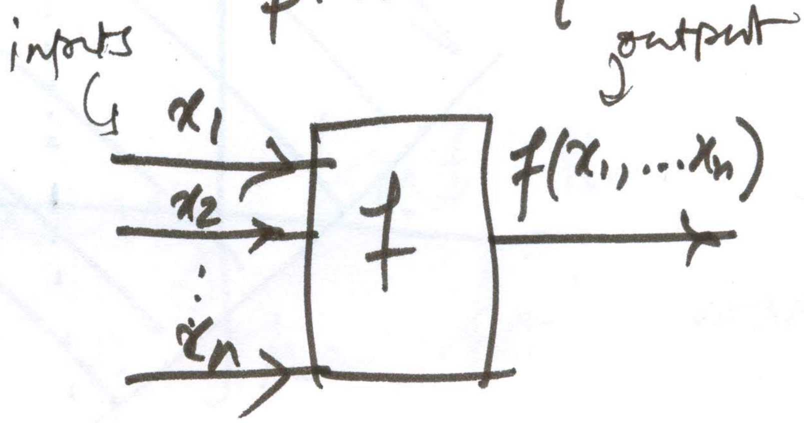
Exp. $\lim_{x \to -\infty} x e^{x}$.

Functions of several variables.

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \ldots \times \mathbb{R}}_{n \text{ time cartesian product of } \mathbb{R}}$$





Sometimes we write

$$z = f(x,y).$$

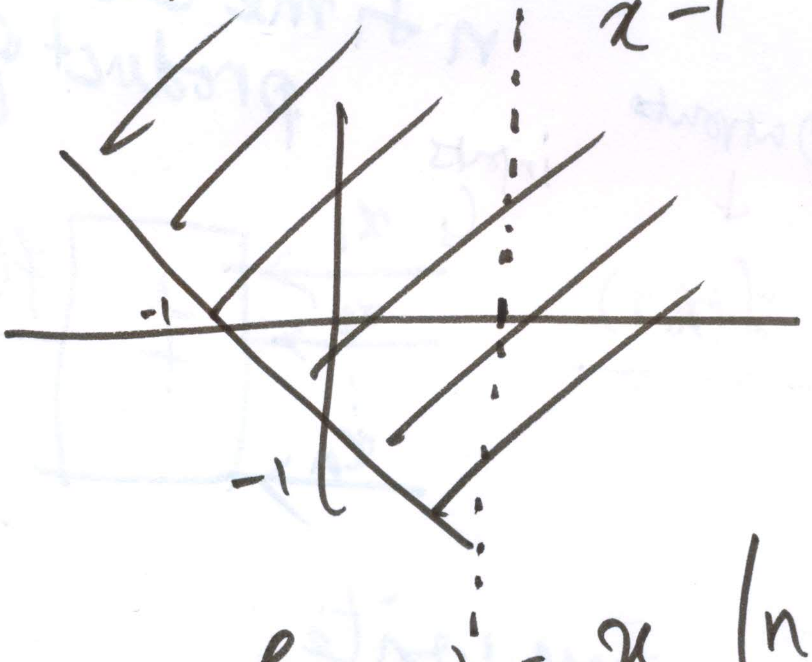
<u>Graph of</u> $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$= \{ f(x,y) \mid x,y \in D \}.$$

Ex1.

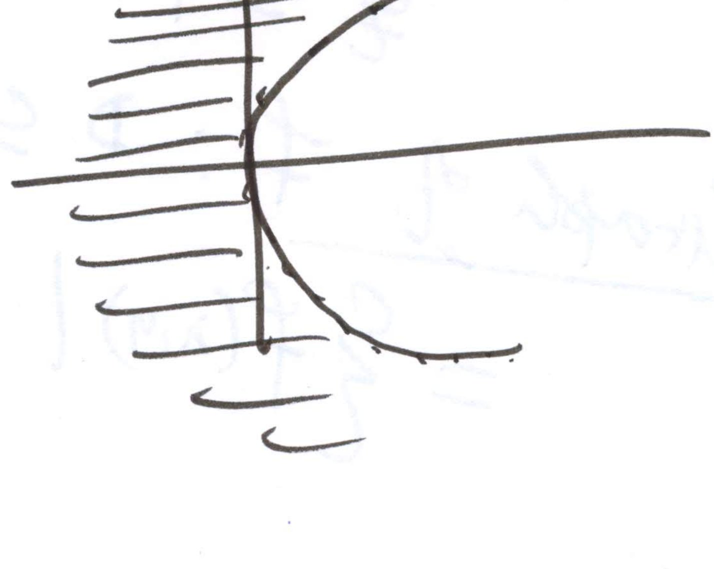
$V = \pi r^2 h = V(r, h)$

Ex2. $f(x, y) = \dfrac{\sqrt{x+y+1}}{x-1}$



← domain

Ex3. $f(x, y) = x \ln(y^2 - x)$



Level curves.

$$f(x,y) = z$$

Then for any fixed K, the curves $f(x,y) = K$ is called a level curve.

It is nothing but the projection of the horizontal plane $z = K$ when it intersected the surface.

Ex.

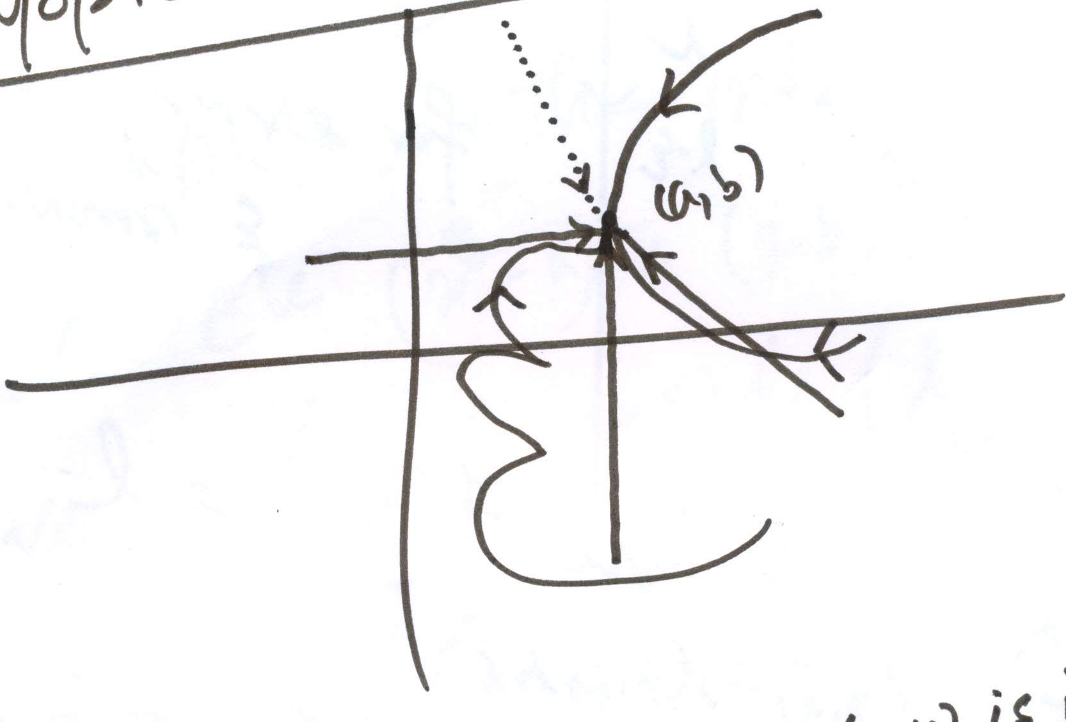
$$f(x,y) = \sqrt{9-x^2-y}$$

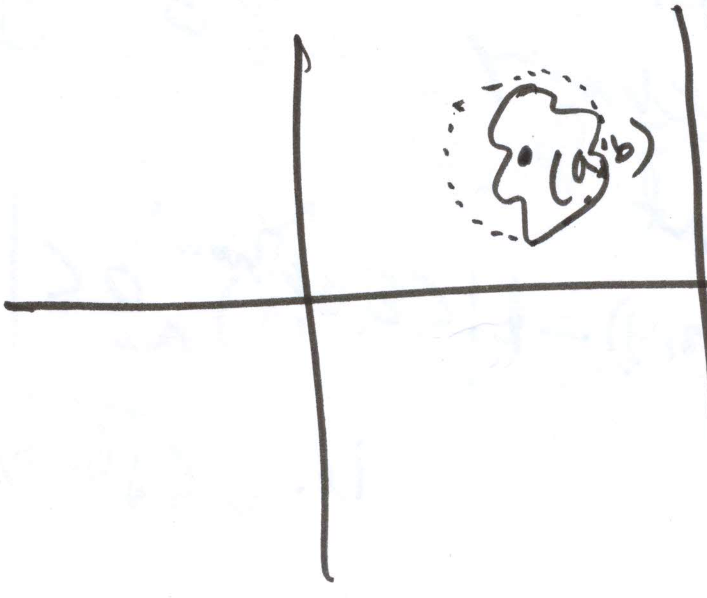
Then $z = K \Rightarrow \sqrt{9-x^2-y^2} = K$

$$\Rightarrow x^2+y^2 = \left(\sqrt{9-K^2}\right)^2$$

$$\lim_{(x,y)\to(a,b)} f(x,y) = l$$

(x,y) approches to (a,b):



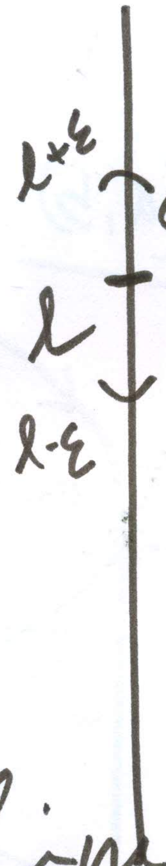


(x,y) is in δ neighborhood of (a,b)

$|(x,y)-(a,b)| < \delta$

for any small $\delta > 0$.

$f(x,y)$ approaches to l



$f(x,y)$ lies in $(l-\varepsilon,\ l+\varepsilon)$

for every $\varepsilon > 0$

ε small.

Def: $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$

for every $\varepsilon > 0$ $\exists\, \delta > 0$

such that

$|f(x,y) - l| < \varepsilon$ when $0 < |(x,y)-(a,b)| < \delta$

i.e. $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$

For an amount ε of change in l, $\exists$ an amount $\delta > 0$ of change of (a,b) such that all inputs in the interior of the circle $(x-a)^2 + (y-b)^2 = \delta^2$ have outputs inside $(l-\varepsilon, l+\varepsilon)$.

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Exp.

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1$$