L-18

Exp 1 Find the general solution of

$$y'' + 2y' + y = x e^{-x}$$

Sol^n.

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{6} x^3 e^{-x}.$$

Exp 2.

$$x^2 y'' - 3xy' + 4y = \ln x, \quad x > 0.$$

Sol:

$$y(x) = c_1 x^2 + c_2 x^2 \ln x + \frac{1}{4} + \frac{1}{4} \ln x, \quad x > 0$$

$$y_1(x) = x^2, \quad y_2(x) = x^2 \ln x, \quad x > 0$$

$$W(x) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix}$$

$$= x^3.$$

$$y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$

$$= -\left(\int \frac{x^2 \ln x}{x^3} dx\right) x^2 + \left(\int \frac{x^2 \ln x}{x^3} dx\right) x^2 \ln x$$

$$= -x^2 \int \frac{(\ln x)^2}{x} dx + x^2 \ln x \int \frac{\ln x}{x} dx$$

$$y(x) = C_1 x^2 + C_2 x^2 \ln x + \frac{x^2 (\ln x)^3}{6}$$

D-operator method.

For ~~so~~ finding P.I. of higher order constant coeffts. ODE.

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^n} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x).$$

$$D = \frac{d}{dx}, \quad D^i = \frac{d^i}{dx^i}$$

$$(a_n D^n + a_n D^{n-1} + \cdots + a_1 D + a_0) y = g(x)$$

input

output.

machine $\quad y = f(x)$

Q. Given output find input.

I. dea

$$\phi(D) = a_n D^n + \cdots + a_1 D + a_0 .$$

The ODE becomes

$$\phi(D)\, y_p = g(x)$$

To ~~bo~~ find $y_p(x)$,

try computing

$$y_p = \phi(D)^{-1} g(x).$$

Exp- $p(D) = D = \frac{d}{dx}$

$y = f(x)$, $D^{-1} f(x) = \int f(x)\, dx$

$p(D) = D^{K}$

$D^{-K} f(x) = \int\int \cdots f(x)\, dx$

$= ?$

Properties-
① $(D^{r} + D^{n})\, y = D^{r} y + D^{n} y$

② $(D^{r} D^{n})\, y = (D^{n} D^{r})\, y$

③ $D^{m}(u + v) = D^{m} u + D^{m} v$

<u>Result 1.</u> $y(x) = e^{ax}$, $\phi(a) \neq 0$

$$\frac{1}{\phi(D)} e^{ax} \cancel{= \frac{1}{\phi(a)} e^{ax}}$$

$$= \frac{1}{\phi(a)} e^{ax}$$

Pf.

$$\frac{1}{\phi(D)} \phi(D) e^{ax} = \frac{1}{\phi(D)} \phi(a) e^{ax}$$

$$\Rightarrow \quad e^{ax} = \frac{1}{\phi(D)} \phi(a) e^{ax}$$

$$\Rightarrow \quad \frac{e^{ax}}{\phi(a)} = \frac{1}{\phi(D)} e^{ax}.$$

$$= \phi(D)^{-1} e^{ax}.$$

Result 2. If $\phi(D) = (D-a)^r \varphi(D)$
and $\varphi(a) \neq 0$

then $$\frac{1}{\phi(D)} e^{ax} = \frac{x^r}{r!\,\varphi(a)} e^{ax}$$

Result 3. $$\frac{1}{\phi(D^2)} \sin(wx+\alpha) = \frac{1}{\phi(-w^2)} \sin(wx+\alpha)$$

where $\phi(-w^2) \neq 0$

and $$\frac{1}{\phi(D^2)} \cos(wx+\alpha) = \frac{1}{\phi(-w^2)} \cos(wx+\alpha)$$

$\phi(-w^2) \neq 0$

Result 4.

$$\frac{1}{p(D)} e^{ax} V(x) = e^{ax} \frac{1}{p(D+a)} V(x)$$

Result 5.

$$\frac{1}{p(D)} x V(x) = \left\{ x - \frac{1}{p(D)} p'(D) \right\} \frac{1}{p(D)} V(x)$$

Result 6. Let $g(x)$ = a polynomial.

(1) ~~[illegible]~~ Let $p(D) = a_n D^n + \cdots + a_1 D + a_0$

If $a_0 \neq 0$ then

$$\frac{1}{p(D)} g(x) = \sum_{j=0}^{\infty} (-1)^j \frac{1}{a_0} \left[\frac{a_n}{a_0} D^n + \cdots + \frac{a_1}{a_0} D \right]^j g(x).$$

② If $a_0 = a_1 = \cdots = a_{k-1} = 0$

$1 \le k \le n$ and $a_k \neq 0$.

then $\dfrac{1}{\phi(D)} g(x)$

$$= \frac{1}{a_n D^n + a_{n-1} D^{n-1} + \cdots + a_k D^k} g(x)$$

$$= \frac{1}{a_n D^{n-k} + a_{n-1} D^{n-k-1} + \cdots + a_k} D^{-k} g(x)$$

then use the previous formula.

Exp. 1 $\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-y$

$= \cos 2x$

Sol^n. P.I. $\phi(D) = D^3+D^2-D-1$

$\phi(D)^{-1} g(x) = \dfrac{1}{D^3+D^2-D-1}\cos 2x$

$= \dfrac{1}{D^2\cdot D+D^2-D-1}\cos 2x$

$= \dfrac{1}{(-4)D+(-4)-D-1}\cos 2x.$

$= -\dfrac{1}{5(D+1)}\cos 2x$

$$= -\frac{D-1}{5(D^2-1)} \cos 2x = -\frac{D-1}{5(-4-1)} \cos 2x$$

$$= -\frac{1}{25}(D-1)\cos 2x$$

$$= +\frac{1}{25}(2\sin 2x + \cos 2x)$$

Exp 2. $\frac{d^3y}{dx^3} + y = \sin 3x + \cos^2\frac{x}{2}$

Sol^n. P.I. $p(D) = D^3 + 1$

$$\frac{1}{D^3+1}\left[\sin 3x + \cos^2\frac{x}{2}\right]$$

$$\frac{1}{D^3+1}\sin 3x + \frac{1}{2}\frac{1}{D^3+1}1$$

$$= \frac{1}{D^3+1}\sin 3x + \frac{1}{2D^3+1}\cos\frac{x}{2}$$

$$= \frac{1}{-9D+1}\sin 3x + \frac{1}{2}\cdot\frac{1}{-D+1}\cos x + \frac{1}{2}(1+D^3)^{-1}1$$

$$= \frac{1}{730}\left(\sin 3x + 27\_\_\right)$$

$$- \frac{1}{2} - \frac{1}{4}\left(\cos x - \sin x\right)$$

<u>Exp.</u> $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y$

$$= x^2 e^x$$

<u>Solⁿ.</u> $\phi(D) = D^3 - 3D + 2$

$$P.I. = \frac{1}{D^3 - 3D + 2} x^2 e^x$$

$$= e^x \frac{1}{(D+1)^3 - 3(D+1) + 2} x^2$$

$$= e^x \frac{1}{D^3 + 3D^2} x^2$$

$$= e^{x} \frac{1}{3D^{2}\left(1+\frac{D}{3}\right)} x^{2}$$

$$= e^{x} \frac{1}{3D^{2}}\left[1+\frac{D}{3}\right]^{-1} x^{2}$$

$$= e^{x} \frac{1}{3D^{2}}\left[1-\frac{D}{3}+\left(\frac{D}{3}\right)^{2}-\cdots\right] x^{2}$$

$$= e^{x} \frac{1}{3D^{2}}\left[x^{2}-\frac{2x}{3}+\frac{2}{3^{2}}\right]$$

$$= \frac{e^{x}}{3D} D^{-1}\left[x^{2}-\frac{2x}{3}+\frac{2}{9}\right]$$

$$= \frac{e^{x}}{3D}\left[\int x^{2}\,dx-\int \frac{2x}{3}\,dx+\int \frac{2}{9}\,dx\right]$$

$$= \frac{e^{x}}{3D}\left[\frac{x^{3}}{3}-\frac{x^{2}}{3}+\frac{2}{9}x\right]$$

$$= \frac{e^{x}}{3}\left[\int \frac{x^{3}}{3}\,dx-\int \frac{x^{2}}{3}\,dx+\int \frac{2}{9}x\,dx\right]$$

$$= \frac{x^2 e^x}{108}\left(3x^2 - 4x + 4\right).$$