Lec 16

Higher order ODE

The general form of a 2^{nd} order LINEAR ODE is

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \quad \text{--- (1)}$$

(a) If $f(x) = 0$ then (1) is called homogeneous

(b) otherwise, non-homogeneous.

The general solution of the 'homogeneous' version of (1) is given by

$$\boxed{y_h(x) = c_1 y_1(x) + c_2 y_2(x)}$$

where $y_1(x)$ & $y_2(x)$ are linearly independent solutions of the homogeneous version

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

where $a_2(x) \neq 0$

Two solutions $Y_1(x)$ & $Y_2(x)$ are linearly independent (L.I.) if

$$C_1 Y_1(x) + C_2 Y_2(x) = 0$$

implies $C_1 = C_2 = 0$.

i.e. $Y_1(x)$ is not a scalar multiple of $Y_2(x)$. or vice-versa.

<u>Remark</u>: If only solution $Y_1(x)$ of a homo. linear 2^{nd} order ODE is known, another L.I. solution can be written as $Y_2(x) = V(x) Y_1(x)$ into the DE. Then solve it to derive $V(x)$.

Exp. Show that $y_1(x) = e^{-2x}$ is a solution of

$$y'' + 4y' + 4y = 0, \quad (1)$$

and find the general solution of this equation.

Sol^n. The general solution of (1) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where $y_1(x) = e^{-2x}$;

$$y_2(x) = x e^{-2x}$$

Linear ODE with constant coefficients.

$$\alpha_n \frac{d^n y}{dx^n} + \alpha_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + \alpha_1 \frac{dy}{dx} + \alpha_0 y = \boxed{0} \quad (*)$$

where $\alpha_n, \ldots, \alpha_0$ are $\boxed{\text{constants}}$.

Let us find general sol$\underline{n}$ of $(*)$.

First let $n=2$

$$a y'' + b y' + c y = 0 \quad (**)$$

Assuming e^{mx} to be a solution of (**) we obtain

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{---} (***)$$

Case I $\quad b^2 - 4ac > 0$

Then the general solution is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

where m_1, m_2 are the roots given by (***).

<u>Case II</u> $b^2 = 4ac$.

Then the general solution of (**) is

$$y = (c_1 + c_2 x) e^{mx}$$

<u>Case III</u> $b^2 - 4ac < 0$

Let $m = k \pm i\omega$ with k, ω real be the roots given by (***).

Then the general sol^n is

$$y = e^{kx}\left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right)$$

Higher order

If the given homo. ODE with constant coeffs. is of order 'n' then the corresponding polynomial equation is

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0. \quad \text{———} (**)$$

Then

Case I. If 'm' is a k-fold/repeated real root then the L.I. of (**) then the soln of $\binom{* \ *}{* \ *}$ are

$e^{m_1 x}, x e^{m_1 x}, x^2 e^{m_1 x},$

$\dots, x^{k-1} e^{m_1 x}$

(2) If $m_1 = a+ib$ and $m_1 = a-ib$ constitute pair of k-fold/repeated pair of complex conjugate roots of $\binom{* \ *}{* \ *}$ then the corresponding L.I. solns are

$e^{ax}\cos bx, \; x e^{ax}\cos bx, \dots, x^{k-1} e^{ax}\cos bx$

$e^{ax}\sin bx, \; x e^{ax}\sin bx, \dots, x^{k-1} e^{ax}\sin bx$

are 2K in number.

Exc. $y^{(4)} - 16y = 0$.

Sol^n.

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x.$$