Lec-14.

If you have not checked your mid sem answer script (or want to recheck), drop me an email after vacation.

'ODE'

F(interms of variables) = 0

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

where y is a function of 'x'.

First order linear ordinary differential equation.

$$\boxed{\frac{dy}{dx} + p(x)\, y = q(x)} \quad \text{--- (1)}$$

Solving (1) means, find a f^n $y(x)$ which satisfies (1).

Separable Form.

$$\frac{dy}{dx} = f(x)\, g(y) \quad \text{———} (*)$$

$$\frac{dy}{g(y)} = f(x)\, dx$$

and integrate

'Homogeneous' 1st order linear ODE.

We call a liner 1st order ODE, a homogeneous equation if it is of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \text{———} (1)$$

i.e. $\frac{dy}{dx}$ is a homogeneous fn of degree zero.

Then (1) can be reduced to $(*)$

Plan

Let $\frac{y}{x} = v(x)$

$\Rightarrow \quad y = x\,v(x)$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Then from (1)

$$v + x\frac{dv}{dx} = f(v)$$

$$\Rightarrow \quad \boxed{\frac{dv}{dx} = \frac{f(v) - v}{x}}$$

Exp. Solve $\frac{dy}{dx} = \frac{x^2 + xy}{xy + y^2}$

Using the procedure

$$-\frac{1}{2}\int \frac{v\,dv}{1-v^2} = \int \frac{dx}{x}$$

$$\Rightarrow \quad -\ln|v| = 2\ln|x| + c_1$$

$$= \ln c_2 x^2 \quad (c_1 = \ln c_2)$$

$$\Rightarrow \quad \frac{1}{|v|} = c_2 x^2 \quad \Rightarrow \quad |1 - v^2| = \frac{c_3}{x^2}$$

where $c_3 = 1/c_2$

$$\Rightarrow \int \frac{v\,dv}{1-v^2} = \int \frac{dx}{x}$$

$$u = 1-v^2 \Rightarrow du = -2v\,dv$$

$$\Rightarrow \frac{1}{|u|} = c_2 x^2$$

$$\Rightarrow |y^2 - x^2| = c$$

$\Rightarrow$ $y^2 - x^2 = c$ is a soln

'Exact' Eqn.

A 1st order diff eqn. expressed in '<u>differential form</u>'

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad (1)$$

which is equivalent to

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \qquad (2)$$

is said to be 'exact' if there exists a fn. $\varphi(x,y)$ such that

$$\underbrace{d\varphi(x,y)}_{\boxed{dz \text{ where } z = \varphi(x,y)}} = M(x,y)\,dx + N(x,y)\,dy$$

Then φ is called the integral fn for (1) and the level curves $\varphi(x,y) = c$ of φ are the solution curves of (1).

The necessary and sufficient condition for (1) to exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad -(3)$$

Q. Why does (3) imply existence of $\varphi(x,y)$.

Exp. Verify that

$$(2x + \sin y - y e^{-x})\,dx + (x\cos y + \cos y + e^{-x})\,dy = 0$$

is exact and find its solution curves.

Sol^n

$$\frac{\partial M}{\partial y} = \cos y - e^{-x} = \frac{\partial N}{\partial x}.$$

Now we want to find $\varphi(x,y)$ such that

$$\frac{\partial \varphi}{\partial x} = M(x,y) = 2x + \sin y - y e^{-x} \quad \text{—(1)}$$

and $$\frac{\partial \varphi}{\partial y} = N(x,y) = x\cos y + \cos y + e^{-x} \quad \text{—(2)}$$

Then from (1), after integration w.r.t. 'x'

$$\varphi(x,y) = x^2 + x\sin y + y e^{-x} + c_1(y)$$

Substituting this expression in (2)

$$c_1'(y) = \cos y \quad \text{—(3)}$$

$\Rightarrow$ $c_1(y) = \sin y + c_2$ after integrating (3) w.r.t y

Then the solution is

$$x^2 + x\sin y + ye^{-y} + \sin y = c$$

for some constant c.

Q. If $M(x,y)\,dx + N(x,y)\,dy = 0$

and it is NOT exact

then how to make it

exact?