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Lec-13

Module-II

Ordinary differential equations (ODE)

Without the knowledge of ODE the study of science ~~of~~ or engineering is almost impossible.

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = y(x)$$

$\frac{dy}{dx}$ → derivative w.r.t. 'x' the independent variable

In real life applications we denote it as $\frac{dx}{dt}$ where $x(t)$ is 'state' of a physical system at time 't' and $\frac{dx}{dt}$ represents evolution of that state.

Exp. (1) Newton's 2nd law of motion

$$\boxed{m \frac{d^2x}{dt^2} = F(t)} \quad \text{---} (*)$$

where $x(t)$ is the position of an object at time t with constant mass 'm' subject to the force $F(t)$, then $x(t)$ must satisfy eq$^{\underline{n}}$ $(*)$

(2) Assuming the biomass $m(t)$ at

$$\boxed{\frac{dm}{dt} = k\, m(t)} \quad \text{---} (**)$$

time 't' of a bacterial culture growing uniformly ~~with~~ infinite resources can be modelled as eq$^{\underline{n}}$ $(**)$

If $k>0$, the growth is exponentially

$k<0$, the decay is exponential

(3) Linear time-invariant systems (LTI)

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$y(t) = D\,x(t) + f(t)$$

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}$$

$\downarrow$ state vector

Differential eqns

- ODE (it involves derivatives w.r.t. only one variable)
- PDE (involves partial derivative of the unknown ft. w.r.t. more than one variable)

<u>Exp.</u> $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

models the lateral displacement $u(x,t)$ at position 'x' at time 't' of a stretched vibrating string.

In order to classify ODE, we first define the order of an ODE, which is the highest-order derivative present in the eqn

Exp-

$$\frac{d^2y}{dx^2} + x^3 y = \sin x \quad \Big| \text{ order } 2$$

$$\frac{d^3y}{dx^3} + 4x\left(\frac{dy}{dx}\right)^2 = y\frac{d^2y}{dx^2} + e^y \quad \Big| \text{ order } 3.$$

n-th order linear ODE:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\, y = f(x) \quad \text{—(***)}$$

where the LHS terms include products of functions of 'x' and 'y' or the derivative of y. There is NO ~~team~~ term as a product of y with derivative of y, or derivative of y with itself.

If RHS $f(x)$ is identically [illegible]
(***) is called homogeneous
otherwise Non-homogeneous

Exp.

① $\frac{d^2y}{dx^2} + x^3 y = \sin x$ | Liner 2nd order non-homo.

② $\frac{d^3y}{dx^3} + 4x\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} - e^y = 0$ | Non-linear

③ $(1+x^2)\frac{d^3y}{dx^3} + \sin x \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$ | Liner 3rd order homogenous.

Thm. If $y = y_1(x)$ & $y = y_2(x)$ are two solutions of

$$a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \dots + a_0 y = 0$$

then $y = A\,y_1(x) + B\,y_2(x)$

For any values of the constants A, B is a solution.

Imp. Thm.

If $y = y_1(x)$ is a soln of a linear homogeneous eqn:

$$a_n(x)\, y^{(n)} + \cdots + a_0(x)\, y = 0 \quad \text{--- (1)}$$

and $y = y_2(x)$ is a solution of the linear non-homogeneous eqn:

$$a_n(x)\, y^{(n)} + \cdots + a_0(x)\, y = f(x) \quad \text{--- (2)}$$

Then $y = y_1(x) + y_2(x)$ is a solution of (2).