Lec-1

Taylor Series representation of a function 'f' at $(a,b) \in D(f)$, the domain of 'f'.

$$f(x,y) \cong f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$+ \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2 \right]$$
$$+ \alpha_0 (\Delta\rho)^3$$

where $\Delta\rho = \sqrt{\Delta x^2 + \Delta y^2}$, $\Delta x = x - a$
$\Delta y = y - b$

and $\Delta\rho \to 0$ when $\Delta x, \Delta y \to 0$

is a 'good' approximation only when (x,y) is close to (a,b).

Lec-12

Taylor series representation of a function 'f' at $(a,b) \in D(f)$, the domain of 'f'.

$$f(x,y) \cong f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2 \right]$$

$$+ \alpha_0 (\Delta\rho)^3$$

where $\Delta\rho = \sqrt{\Delta x^2 + \Delta y^2}$, $\Delta x = x-a$
$\Delta y = y-b$

and $\Delta\rho \to 0$ when $\Delta x, \Delta y \to 0$

is a 'good' approximation only when (x,y) is close to (a,b).

If (a,b) attaeins a local maxima of 'f'

$$f(x,y) \le f(a,b) \text{ when } (x,y) \to (a,b)$$

and (a,b) attaim a local minima of 'f' ~~at~~ then

$$f(x,y) \ge f(a,b) \text{ when } (x,y) \to (a,b).$$

Let ~~x~~ $y = b$ and consider the f: $f(x,b)$ then $f($~~x~~$,b)$ has a local max/min at $x=a$.

llly, if $x=a$ then $f(a,y)$ has a local max/min at $y=b$.

Then from Fermat's theorm.

$$\frac{\partial f}{\partial x}(x,b) = 0 \text{ at } x=a$$

$$\Rightarrow f_x(a,b) = 0$$

llly, $\frac{\partial f}{\partial y}(a,y) = 0$ at $y=b$ $\Rightarrow$ $f_y(a,b)=0$

Thm. A f^n f has local max or min at (a,b) then
$$f_x(a,b) = 0 = f_y(a,b).$$

Critical point: A point (a,b) is called a critical pt. for the f^n f if $f_x(a,b) = 0$ and $f_y(a,b) = 0$ or if one of these partial derivatives does not exist.

Exp. $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

The critical point is $(1,3)$

Exp. $f(x,y) = y^2 - x^2$

The critical pt. is $(0,0)$

Characterization of a critical point

Thm. Let a fn. $f(x,y)$ have continuous partial derivatives up to order 3 ~~in a~~ near a point (a,b). In addition, let the pt. (a,b) be a critical pt. of $f(x,y)$ i.e.

$$\frac{\partial f}{\partial x}(a,b) = 0, \ \frac{\partial f}{\partial y}(a,b) = 0.$$

$$\text{Let } D = f_{xx}(a,b)\, f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2.$$

Then at (a,b)

① $f(x,y)$ has a local max if $D > 0$ and $f_{xx}(a,b) < 0$

② $f(x,y)$ has a local min if $D > 0$ and $f_{xx}(a,b) > 0$

③ $f(x,y)$ has neither local max nor local min, and (a,b) is called a saddle pt. if $D < 0$.

④ If $D = 0$ there may or may not be a local extremum.

Exp. Characterize critical points of

$f(x,y) = x^3 + y^3 - 3xy$

max/min $f(x,y,z)$

subject to some constraints

$$g(x,y,z) = K_1$$
$$h(x,y,z) = K_2$$

method of Lagrange multipliers.

max/min $f(x,y,z)$

s.t. $g(x,y,z) = K$

Assume that the extreme values exist and $\nabla g = \left[\frac{\partial g}{\partial x} \; \frac{\partial g}{\partial y} \; \frac{\partial g}{\partial z}\right]$ $\neq 0$ on the surface $g(x,y,z) = K$

1. Find all values of (x,y,z) and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

and $g(x,y,z) = K$

(2) Evaluate f at all the points obtained from (1). The largest of these values is the maximum value of 'f', the smallest is the min value of 'f'.

Ex. max/min $f(x,y) = x^2 + 2y^2$

s.t. $g(x,y) = x^2 + y^2 = 1$

Sol^n. $2x = \lambda 2x, \quad (1) \qquad 4y = \lambda 2y, \quad (2)$

$x^2 + y^2 = 1 \quad (3)$

From (1) $\lambda = 1$ or $x = 0$

If $x = 0$ then from (3) $y = \pm 1$

If $\lambda = 1$ then $y = 0$ from eq(2) which gives $x = \pm 1$ from (3)

Hence the possible values of 'f' are at the points $(0,1)$, $(0,-1)$, $(1,0)$ and $(-1,0)$

Exp. Find max $f(x,y,z) = x + 2y + 3z$

s.t. $x - y + z = 1$

and $x^2 + y^2 = 1$

Soln $\nabla f = \lambda \nabla g + \mu \nabla h$.

where $g = (x - y + z) = 1$

$h = (x^2 + y^2 = 1$.

i) $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} + \mu \frac{\partial h}{\partial x}$

$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} + \mu \frac{\partial h}{\partial y}$

$\frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} + \mu \frac{\partial h}{\partial z}$

Ans. $3 + \sqrt{29}$

Find absolute max / min of a continuous fn 'f' on a closed and bounded set D:

1. Find the values of 'f' at the critical points of 'f' in D.
2. Find the extreme values of 'f' on the boundary of D
3. The largest of the values from step 1 and 2 is the absolute max value the smallest of these values is the absolute min. value.

<u>Exp.</u> Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \le 1$