Lec-10

Taylor series expansion for functions of two variables.

$$f : D \subseteq \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

approximation of 'f' using a polynomial of 'two' variables at around/near a point $(a,b) \in D$.

Recall $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$

at around 'a' $\in D$.

$$f(x) \cong f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi).$$

$$\xi = a + \theta(x-a)$$
$$0 < \theta < 1$$

Let $z = f(x,y)$ which is continuous (no hole/break), together with all its partial derivatives up to order $(n+1)$ in some neighborhood of (a,b)/ near (a,b)/ around (a,b)/ about (a,b).

Let x to be constant. Then the Taylor series expansion of $f(x,y)$ around/about/near $y=b$ is as follows.

~~$f(x,y) =$~~

$$f(x,y) = f(x,b) + \frac{\partial f}{\partial y}(x,b)\,(y-b) + \frac{\partial^2 f}{\partial y^2}(x,b)\,\frac{(y-b)^2}{2!} + \cdots + \frac{(y-b)^3}{3!}\,\frac{\partial^3 f}{\partial y^3}(x,\xi) \qquad \text{——— (1)}$$

<u>Note</u>: Consider terms in the Taylor series of $f(x,b)$, $\frac{\partial f}{\partial y}(x,b)$, $\frac{\partial^2 f}{\partial y^2}(x,b)$ up to $n=2$.

~~However~~ the Taylor series expansion of $f(x,b)$ at arround ~~a~~ '$x=a$' is as follows.

$$f(x,b) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial^2 f}{\partial x^2}(a,b)\frac{(x-a)^2}{2!} + \frac{\partial^3 f}{\partial x^3}(\xi_1, b)\frac{(x-a)^3}{3!} \quad \text{---} (2)$$

$\xi_1 = a + \theta_1 (x-a)$, $0<\theta_1<1$

The Taylor series expansion of $\frac{\partial f}{\partial y}(x,b)$ at arround '$x=a$' is as follows.

$$\frac{\partial f}{\partial y}(x,b) = \frac{\partial f}{\partial y}(a,b) + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(a,b)(x-a) + \frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial y}(\xi_2, b)\frac{(x-a)^2}{2!}$$

where $\xi_2 = a + \theta_2 (x-a)$, $0<\theta_2<1$

$$= f_y(a,b) + f_{xy}(a,b)(x-a) + f_{xxy}(\xi_2, b)\frac{(x-a)^2}{2!} \quad \text{---} (3)$$

The Taylor series expansion of $\frac{\partial^2 f}{\partial y^2}(x,b)$ at arround '$x=a$'

$$\frac{\partial^2 f}{\partial y^2}(x,b) = f_{yy}(a,b) + f_{xyy}(\xi_3, b)(x-a) \quad \text{---} (4)$$

Substituting (2), (3), (4) in (1)
we obtain

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2!}\Big[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2 \Big]$$

$$+ \frac{1}{3!}\Big[f_{xxx}(\xi_1,b)(x-a)^3 + 3f_{xxy}(\xi_2,b)(x-a)^2(y-b) + 3f_{xyy}(\xi_3,\xi_2)(x-a)(y-b)^2 + f_{yyy}(x,\xi)(y-b)^3 \Big]$$

R_2

Reminder.

Let $x-a = \Delta x$, $y-b = \Delta y$, $\Delta\rho = \sqrt{\Delta x^2 + \Delta y^2}$

Then $|\Delta x| < \Delta\rho$, $|\Delta y| < \Delta\rho$

$$R_2 = \frac{1}{3}\underbrace{\Big[f_{xxx}(\xi_1,b)\frac{\Delta x^3}{\Delta\rho^3} + \cdots + f_{yyy}(x,\xi)\frac{\Delta y^3}{\Delta\rho^3} \Big]}_{\alpha} \Delta\rho^3 < \alpha \times \Delta\rho^3 < \infty$$

Exp.

$$f(x,y) = e^{x} \sin(x-y)$$

Find a Taylor series representation of $f(x,y)$ up to order 2 at $(0,0)$.

Ans. $f(x,y) = x^2 + x - y - xy$

Approximating $f(a+h, b+k)$.
using Taylor series.

Suppose $f(0.1, -0.1)$ for
$f(x,y) = e^{x} \sin(x-y)$.

Using the Taylor series approximation of 'f' around $(0,0)$, we can approximate $f(0.1, -0.1)$ without evaluating $f(0.1, -0.1)$ directly.