Lec-9

# Derivative of implicit functions.

$y = f(x)$

$\frac{dy}{dx} = \checkmark$

y is written, explicitely in terms of 'x'

Exp. $x^3 + y^3 = 6xy$

$\Rightarrow y^3 + x^3 - 6xy = 0$

$\Rightarrow \boxed{F(x,y) = 0}$

Thm. Let $y$ be a continuous fn of $x$ defined implicitely by $F(x,y) = 0$ where $F(x,y)$, $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ are continuous functions in some domain D containing the point $(x,y)$ and $\frac{\partial F}{\partial y}(x,y) \neq 0$.

Then $\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}$

Pf.

$$F(x+\Delta x, y+\Delta y) - F(x,y) \cong 0$$

where $\Delta x, \Delta y \to 0$

$$\Rightarrow \frac{\partial F}{\partial x}(x,y)\,\Delta x + \frac{\partial F}{\partial y}(x,y)\,\Delta y$$

$$+ \varepsilon_1 \Delta x + \varepsilon_2 \Delta y = 0$$

where $\varepsilon_1, \varepsilon_2 \to 0$ when $\Delta x, \Delta y \to 0$

$$\Rightarrow \frac{\partial F}{\partial x}(x,y) + \frac{\partial F}{\partial y}(x,y)\,\frac{\Delta y}{\Delta x}$$

$$+ \varepsilon_1 + \varepsilon_2 \frac{\Delta y}{\Delta x} = 0$$

letting $\Delta x \to 0$

$$\Rightarrow \frac{\partial F}{\partial x}(x,y) + \frac{\partial F}{\partial y}(x,y)\,\frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}}$$

Ans. $$-\frac{x^2-2y}{y^2-2x}.$$

Let us assume that

$$F(x, y, z) = 0$$

where $z = f(x, y)$

Exp. $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$

using chain rule

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$ since $\frac{\partial y}{\partial x} = 0$ as $y$ is independent of '$x$'.

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}}$$

similarly $\boxed{\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}}$

$$z = f(x,y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} \text{ or } \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}$$

1) Total differential is important!

There is an alternative way to compute the total differential of a fn 'f' at (0,0).

Let us introduce a class of functions.

## Homogeneous functions in 'n' variables $x_1, x_2, \ldots, x_n$

$f(x_1, x_2, \ldots, x_n)$ is said to be a homogeneous function of 'degree' $K$ if

$$f(tx_1, tx_2, \ldots, tx_n) = t^{K} f(x_1, x_2, \ldots, x_n)$$

where $t \in \mathbb{R}$

Exp. (1) $f(x_1, x_2, x_3) = x_1^2 x_2^3 x_3$

**Exp.**

$$f(x_1, x_2) = \sqrt{x_1^3 + x_2^3}$$

$$f(tx_1, tx_2) = \sqrt{t^3 x_1^3 + t^3 x_2^3}$$

$$= t^{3/2}\sqrt{x_1^3 + x_2^3}$$

$$= t^{3/2} f(x_1, x_2)$$

degree.

**Observation.** $f(x) = \alpha x^K$

and $f(tx) = t^m f(x)$

Q. Find a homogeneous fn of one variable that is not of the type

$$f(tx) = \alpha (tx)^K = t^K \alpha x^K$$

$$= t^K f(x)$$

Then $$\frac{df}{dx} = \alpha k\, x^{k-1} = \alpha k \frac{x^k}{x}$$

$$\Rightarrow \boxed{x \frac{df}{dx} = \alpha k\, x^k = k f(x)}$$

Obs. $$x f'(x) = k f(x)$$

i.e. we do not need to compute $f'(x)$ but it can be obtained from $f(x)$.

Euler's theorem for homogeneous fn. two-variables

Let $z = f(x,y)$ be a homogeneous fn of degree '$k$'. Then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = k f(x,y).$$

Pf. Let $X = xt$, $Y = yt$

Then $f(X, Y) = t^k f(x,y)$

Now differentiate w.r.t. '$t$'.

$$\Rightarrow \frac{\partial f}{\partial t} = \frac{\partial f}{\partial X}\frac{\partial X}{\partial t} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial t}$$

$$= x\frac{\partial f}{\partial X} + y\frac{\partial f}{\partial Y}$$

However, $\frac{\partial f}{\partial t} = K\,t^{K-1} f(x,y)$.

Now at $t=1$,

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = K f(x,y)}$$

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Q. $X = xt$

$$\frac{\partial f}{\partial X} = \frac{\partial f}{\partial x} \text{ at } t=1 ?$$

$$\frac{\partial f}{\partial X} = \frac{\partial f}{t\,\partial x} \text{ at } t\neq 1$$

Any homogeneous function 'f' of two variables of degree 'k' can also be written as

$$f(x,y) = x^k \phi\left(\frac{y}{x}\right)$$

or $f(x,y) = y^k \phi\left(\frac{x}{y}\right)$

for some function $\phi$

Prove Euler's theorem for

Exp! Let $f(x,y) = x^2 y^3$

$$= x^5 \left(\frac{y}{x}\right)^3$$

$$f(x,y) = y^5 \left(\frac{x}{y}\right)^2$$

Exp. 2.

Let $f(x,y) = x^k \varphi\left(\frac{y}{x}\right)$

Then prove That

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = k\, f(x,y)$$

Pf. H.W.

Exp. 1. Let $f(x,y) = \dfrac{\sqrt{x} + \sqrt{y}}{x+y}$

Compute $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}$.

Ans. $-\dfrac{1}{2}\,\dfrac{\sqrt{x}+\sqrt{y}}{x+y}$

$$= x^{-1/2}\,\frac{x+\sqrt{xy}}{x+y}$$

$$= x^{-1/2}\,\frac{1+\sqrt{y/x}}{1+y/x} = x^{-1/2}\,\varphi\left(\frac{y}{x}\right)$$

Enter's thm.

Let $z = f(x,y)$ be a homogeneous $f^n$ of degree $K$. Then

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = K(K-1) f(x,y)$$

Hint. ~~A~~ssume $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Pf. $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = K f(x,y)$ —(1)

Differentiate (1) ~~[illegible]~~ w.r.t. '$x$' —(2)

(1) w.r.t. '$y$' —(3)

Then $x \times (2) + y \times (3)$ gives the desired result.

Exp. 1 If $z = e^{x/y}\sin\left(\frac{x}{y}\right) + e^{y/x}\cos\left(\frac{y}{x}\right)$

Prove $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

Exp 2. If $z = \left(\frac{x}{y}\right)^{y/x}$

Prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$$

Exp 3. If $z = x^n f\left(\frac{y}{x}\right) + y^{-n} f\left(\frac{x}{y}\right)$

Prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z.$$

Solⁿ. Let $z = u + v$

where $u(x,y) = x^n f(y/x)$ is a homogeneous f: of degree 'n'

& $v(x,y) = y^{-n} f(x/y)$ is a homogeneous f: of degree '-n'.

By Euler's theorem.

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

and $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

Adding both.

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n^2 u \quad \text{——} (1)$$

Similarly for $v(x,y)$.

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + \dots \quad \text{——} (2)$$

Then (1) + (2) gives the desired result.

~~Xt = xt~~

## Euler's theorem for '3' variables

Let $f(x,y,z)$ be a homogeneous fn of degree $K$. Then

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=K f(x,y,z)$$

$$X = xt, \quad Y = yt, \quad Z = zt$$

Pf. Then $f(X,Y,Z) = t^K f(x,y,z)$.

Differentiating w.r.t. 't'

$$\frac{\partial f}{\partial X}\frac{\partial X}{\partial t}+\frac{\partial f}{\partial Y}\frac{\partial Y}{\partial t}+\frac{\partial f}{\partial Z}\frac{\partial Z}{\partial t}=K t^{K-1} f(x,y,z).$$

$$\Rightarrow x\frac{\partial f}{\partial X}+y\frac{\partial f}{\partial Y}+z\frac{\partial f}{\partial Z}=K t^{K-1} f(x,y,z)$$

Setting $t=1$

$$\Rightarrow x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=K f(x,y,z)$$