GIAN course: Singular optimal control

Slide collection 9

Pseudospectra for Structured Matrix Perturbations

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Motivation.

Situation:

The entries of a matrix $\mathcal{A} \in \mathbb{C}^{n\times n}$ have been obtained by measurement

$$\mathcal{A} = A + \Delta, \qquad \|\Delta\| \le \rho$$

$\mathcal{A}$: true value of $\mathcal{A}$

A: measured value

Δ: measurement error

ρ: error bound

Alternative point of view: Regard Δ as a perturbation of A.

Problem: What is the spectrum of $\mathcal{A}$?

Well known:

The spectrum of a matrix can depend very sensitively on perturbations.

Example:

$$\underbrace{\begin{bmatrix} 2 & 10^{k+1} \\ 10^{-k} & 5 \end{bmatrix}}_{\mathcal{A}_k} = \underbrace{\begin{bmatrix} 2 & 10^{k+1} \\ 0 & 5 \end{bmatrix}}_{A_k} + \underbrace{\begin{bmatrix} 0 & 0 \\ 10^{-k} & 0 \end{bmatrix}}_{\Delta_k}, \qquad k \in \mathbb{N}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

Spectra: $\sigma(\mathcal{A}_k) = \{0, 7\}$, $\sigma(A_k) = \{2, 5\}$

First order Taylor-Expansion of Eigenvalues of A_k yields the estimate

$$\sigma(\mathcal{A}_k) \approx \{-1.3,\ 8.3\}$$

Analytic pert. theory: Taylor-Expansion of a simple eigenvalue

Let $\lambda_0 \in \mathbb{C}$ be a simple eigenvalue of $A \in \mathbb{C}^{n\times n}$ with right eigenvector x (column vector) and left eigenvector y^* (row vector), i.e.

$$Ax = \lambda_0\, x, \qquad y^* A = \lambda_0\, y^*.$$

If $\Delta \in \mathbb{C}^{n\times n}$ is sufficiently small than $A + \Delta$ has a simple eigenvalue $\lambda(A+\Delta)$ close to λ_0 and

$$\lambda(A+\Delta) \;=\; \lambda_0 + \frac{y^*\Delta x}{y^* x} + \mathcal{O}(\|\Delta\|^2)$$

Analytic pert. theory: Taylor-Expansion of a simple eigenvalue

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$$Ax = \lambda_0 \, x, \qquad y^* A = \lambda_0 \, y^*.$$

If $\Delta \in \mathbb{C}^{n\times n}$ is sufficiently small than $A + \Delta$ has a simple eigenvalue $\lambda(A+\Delta)$ close to λ_0 and

$$\begin{aligned}\lambda(A+\Delta) &= \lambda_0 + \frac{y^*\Delta x}{y^* x} + \mathcal{O}(\|\Delta\|^2) \\ &= \lambda_0 + \frac{y^*\Delta x}{y^* x} + \frac{y^*\Delta(\lambda_0 I - A)^D \Delta x}{y^* x} + \mathcal{O}(\|\Delta\|^3)\end{aligned}$$

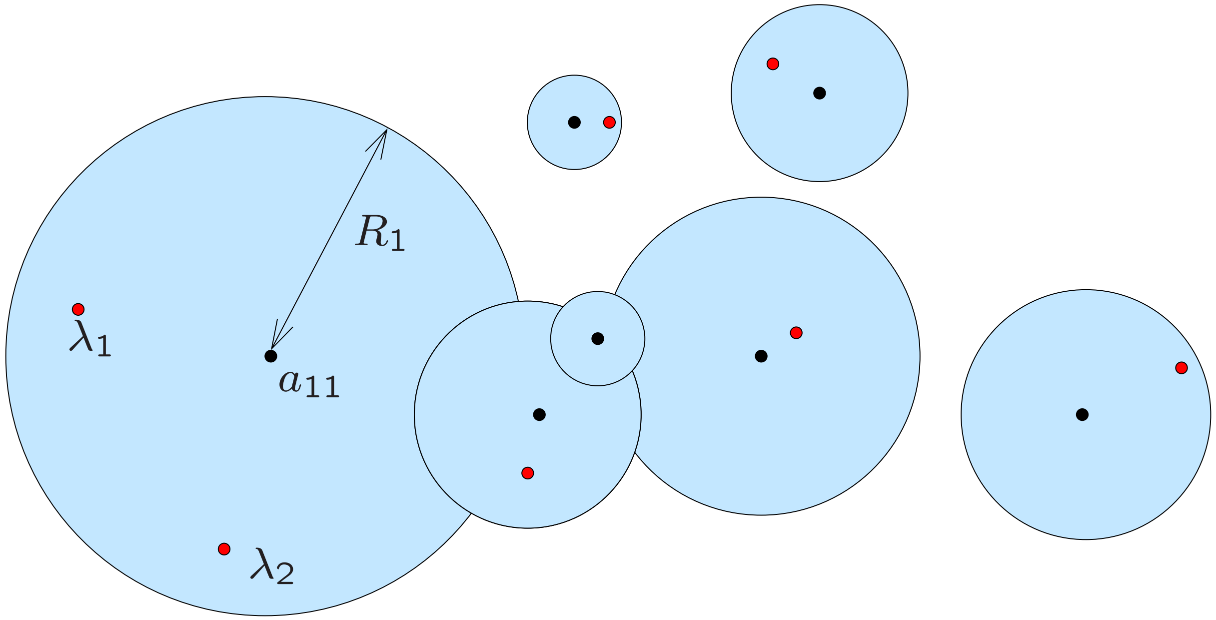
Here M^D denotes the **Drazin inverse** which is defined by $M^D v = 0$ for all $v \in \ker M^n$ and

$$M^D M v = v \quad \text{for all} \quad v \in \bigoplus_{\lambda\in\sigma(M)\setminus\{0\}} \ker(\lambda I - M)^n$$

Eigenvalue inclusion theorems: Gershgorin (1931)

All eigenvalues of $A = [a_{jk}] \in \mathbb{C}^{n\times n}$ are contained in the union of the discs

$$D(a_{jj}, R_j) = \{z \in \mathbb{C};\ \ |z - a_{jj}| \le R_j\}, \qquad R_j = \sum_{k\neq j} |a_{jk}|, \qquad k = 1, \dots, n$$



Structured Pseudospectra

Data:

Nominal matrix	:	$A \in \mathbb{C}^{n\times n}$
Perturbation class	:	$\text{struct} \subseteq \mathbb{C}^{n\times n}$ subspace (over $\mathbb{C}$ or $\mathbb{R}$)
$\|\cdot\|$	:	norm on struct
Perturbation level	:	$\rho \geq 0$.

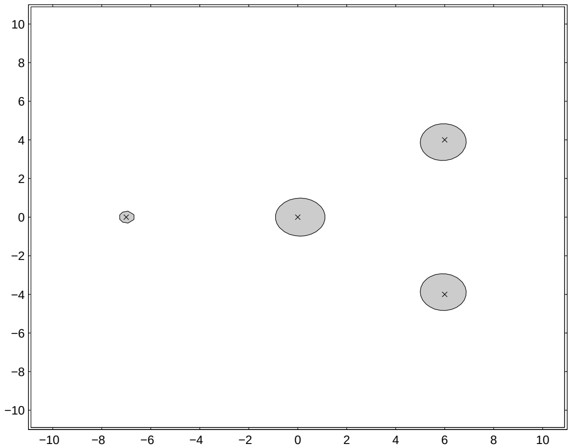
Definition:

$$\sigma_{\text{struct}}(A,\rho) \quad := \quad \text{Set of eigenvalues of all matrices of the form}$$

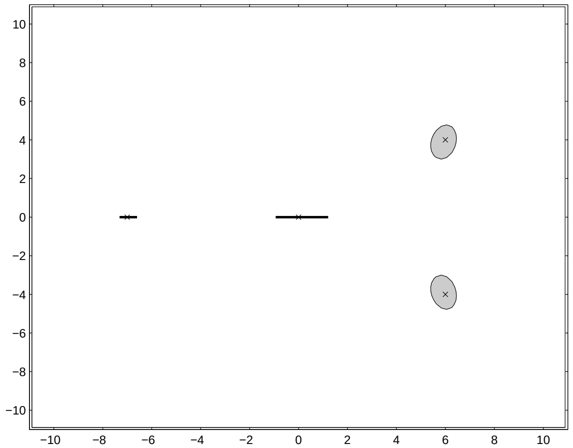
$$A + \Delta, \qquad \Delta \in \text{struct},\ \|\Delta\| \leq \rho.$$

Example:

$\rho = .05$



$\text{struct} = \mathbb{C}^{n\times n}$



$\text{struct} = \mathbb{R}^{n\times n}$

Structured Pseudospectra

Data:

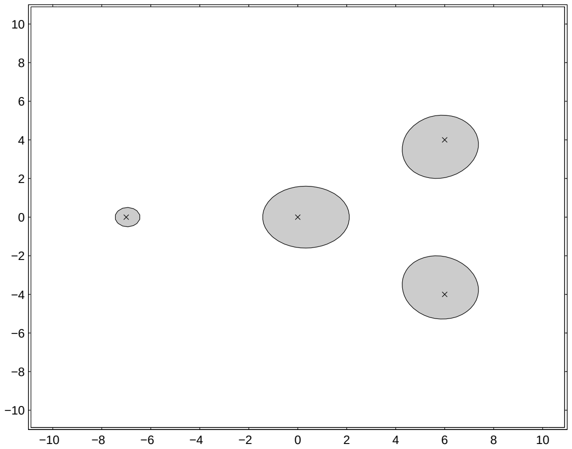
Nominal matrix	:	$A \in \mathbb{C}^{n\times n}$
Perturbation class	:	$\mathsf{struct} \subseteq \mathbb{C}^{n\times n}$ subspace (over $\mathbb{C}$ or $\mathbb{R}$)
$\|\cdot\|$	:	norm on struct
Perturbation level	:	$\rho \geq 0$.

Definition:

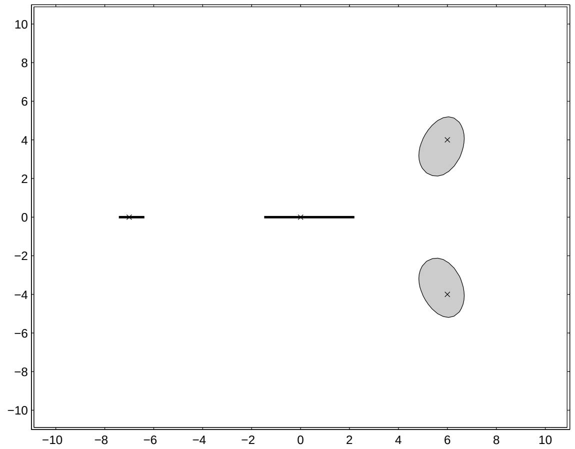
$$\sigma_{\mathsf{struct}}(A,\rho) \;:=\; \text{Set of eigenvalues of all matrices of the form } A+\Delta, \quad \Delta \in \mathsf{struct},\ \|\Delta\| \leq \rho.$$

Example:

$\rho = .08$



$\mathsf{struct} = \mathbb{C}^{n\times n}$



$\mathsf{struct} = \mathbb{R}^{n\times n}$

Structured Pseudospectra

Data:

Nominal matrix	:	$A \in \mathbb{C}^{n\times n}$
Perturbation class	:	$\text{struct} \subseteq \mathbb{C}^{n\times n}$ subspace (over $\mathbb{C}$ or $\mathbb{R}$)
$\|\cdot\|$	:	norm on struct
Perturbation level	:	$\rho \geq 0$.

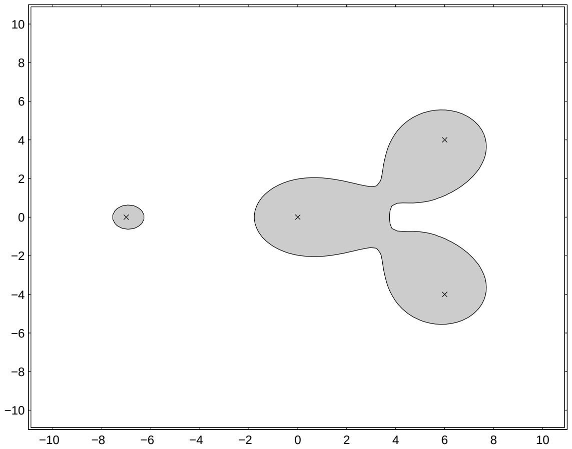
Definition:

$\sigma_{\text{struct}}(A, \rho)$:= Set of eigenvalues of all matrices of the form

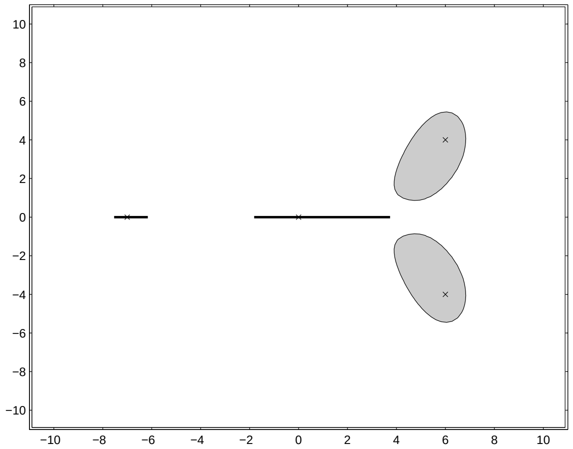
$$A + \Delta, \qquad \Delta \in \text{struct}, \ \|\Delta\| \leq \rho.$$

Example:

$\rho = 1.0$



$\text{struct} = \mathbb{C}^{n\times n}$



$\text{struct} = \mathbb{R}^{n\times n}$

Structured Pseudospectra

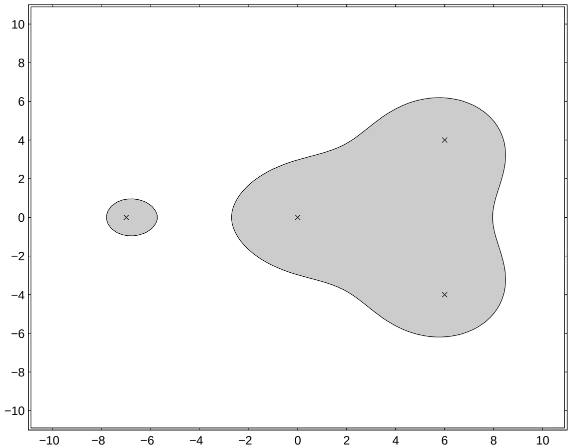
Data:	Nominal matrix	:	$A \in \mathbb{C}^{n\times n}$
	Perturbation class	:	$\mathsf{struct} \subseteq \mathbb{C}^{n\times n}$ subspace (over $\mathbb{C}$ or $\mathbb{R}$)
	$\|\cdot\|$	:	norm on struct
	Perturbation level	:	$\rho \geq 0$.

Definition:

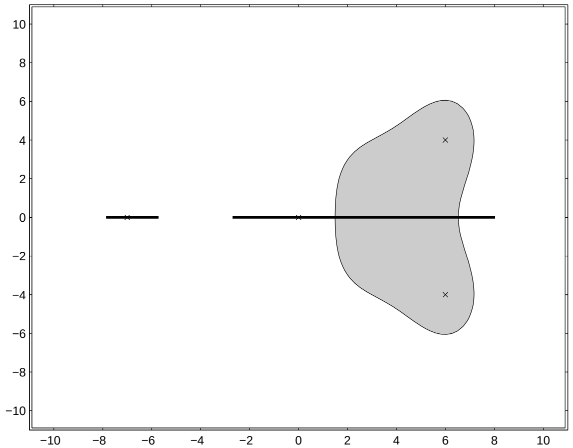
$$\sigma_{\mathsf{struct}}(A,\rho) := \text{Set of eigenvalues of all matrices of the form } A+\Delta, \quad \Delta \in \mathsf{struct},\ \|\Delta\| \leq \rho.$$

Example:

$\rho = 1.5$



$\mathsf{struct} = \mathbb{C}^{n\times n}$



$\mathsf{struct} = \mathbb{R}^{n\times n}$

Structured Pseudospectra

Data:

Nominal matrix	:	$A \in \mathbb{C}^{n\times n}$
Perturbation class	:	struct $\subseteq \mathbb{C}^{n\times n}$ subspace (over $\mathbb{C}$ or $\mathbb{R}$)
$\|\cdot\|$	:	norm on struct
Perturbation level	:	$\rho \geq 0$.

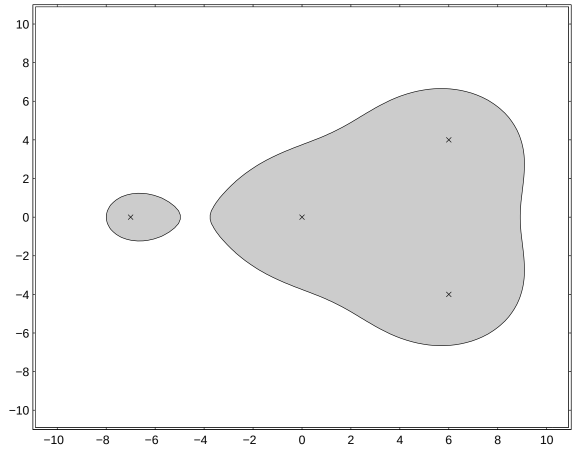
Definition:

$\sigma_{\text{struct}}(A,\rho)$:= Set of eigenvalues of all matrices of the form

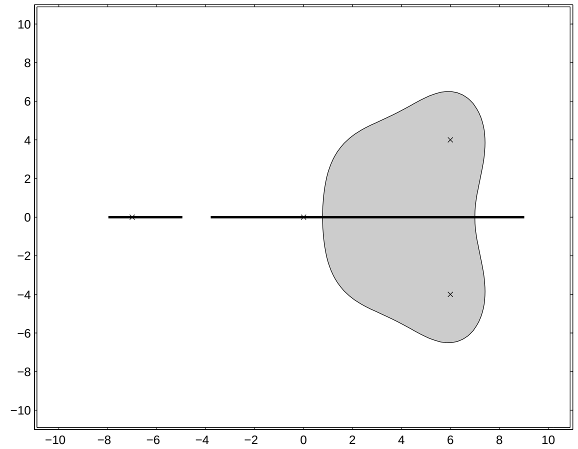
$A + \Delta, \qquad \Delta \in \text{struct}, \ \|\Delta\| \leq \rho.$

Example:

$\rho = 1.9$



struct $= \mathbb{C}^{n\times n}$



struct $= \mathbb{R}^{n\times n}$

Structured Pseudospectra

Data:

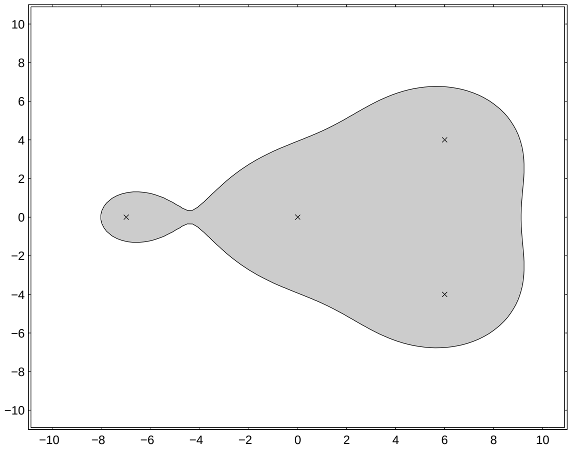
Nominal matrix	:	$A \in \mathbb{C}^{n\times n}$
Perturbation class	:	struct $\subseteq \mathbb{C}^{n\times n}$ subspace (over $\mathbb{C}$ or $\mathbb{R}$)
$\|\cdot\|$	:	norm on struct
Perturbation level	:	$\rho \geq 0$.

Definition:

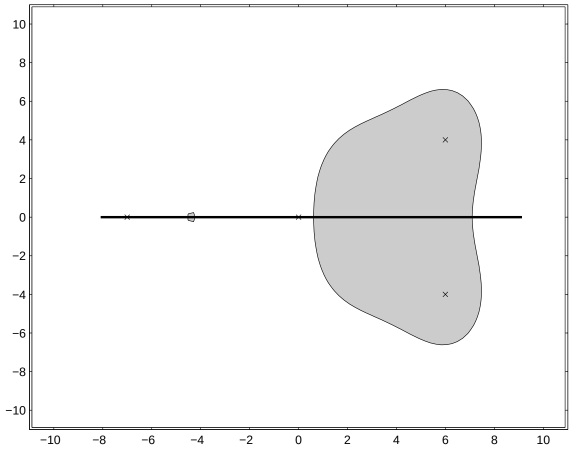
$$\sigma_{\text{struct}}(A,\rho) \;:=\; \text{Set of eigenvalues of all matrices of the form } A+\Delta, \qquad \Delta \in \text{struct},\ \|\Delta\| \leq \rho.$$

Example:

$\rho = 2.0$



struct $= \mathbb{C}^{n\times n}$



struct $= \mathbb{R}^{n\times n}$

Structured Pseudospectra

Data:

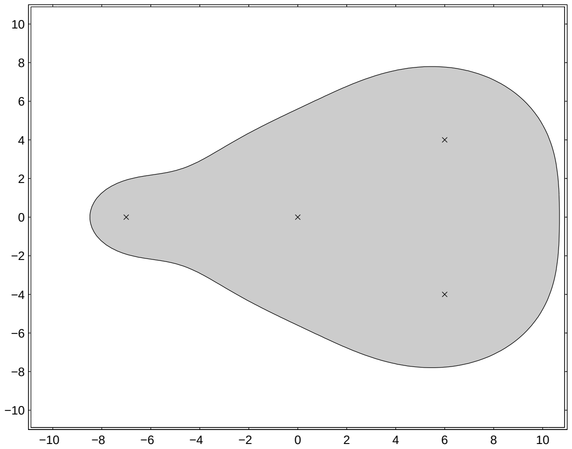
Nominal matrix	:	$A \in \mathbb{C}^{n\times n}$
Perturbation class	:	$\text{struct} \subseteq \mathbb{C}^{n\times n}$ subspace (over $\mathbb{C}$ or $\mathbb{R}$)
$\|\cdot\|$	:	norm on struct
Perturbation level	:	$\rho \geq 0$.

Definition:

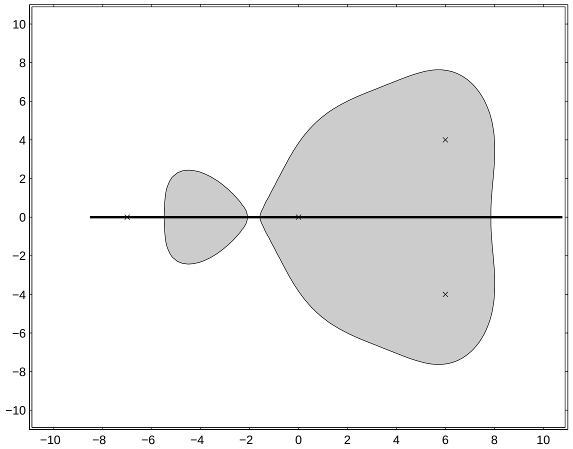
$$\sigma_{\text{struct}}(A,\rho) \quad := \quad \text{Set of eigenvalues of all matrices of the form}$$
$$A + \Delta, \qquad \Delta \in \text{struct}, \ \|\Delta\| \leq \rho.$$

Example:

$\rho = 3.0$



$\text{struct} = \mathbb{C}^{n\times n}$



$\text{struct} = \mathbb{R}^{n\times n}$

Structured Pseudospectra

Data:	Nominal matrix	:	$A \in \mathbb{C}^{n\times n}$
	Perturbation class	:	$\mathsf{struct} \subseteq \mathbb{C}^{n\times n}$ subspace (over $\mathbb{C}$ or $\mathbb{R}$)
	$\|\cdot\|$	:	norm on struct
	Perturbation level	:	$\rho \geq 0$.

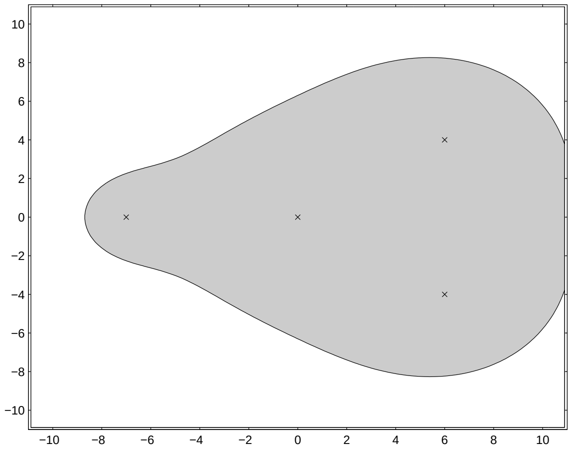
Definition:

$$\sigma_{\mathsf{struct}}(A,\rho) \quad := \quad \text{Set of eigenvalues of all matrices of the form}$$

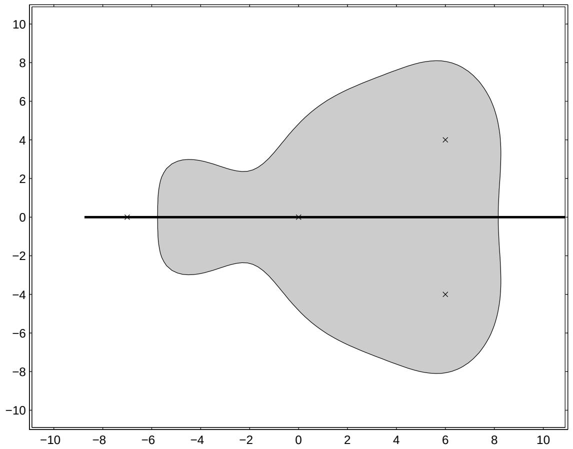
$$A + \Delta, \qquad \Delta \in \mathsf{struct},\ \|\Delta\| \leq \rho.$$

Example:

$\rho = 3.5$



$\mathsf{struct} = \mathbb{C}^{n\times n}$



$\mathsf{struct} = \mathbb{R}^{n\times n}$

Structured Pseudospectra

Data:

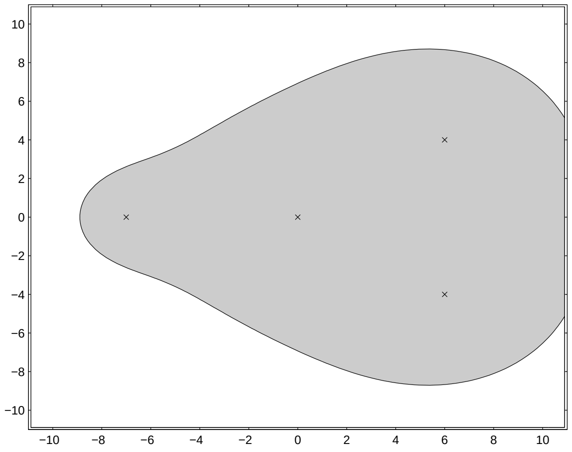
Nominal matrix	:	$A \in \mathbb{C}^{n\times n}$
Perturbation class	:	struct $\subseteq \mathbb{C}^{n\times n}$ subspace (over $\mathbb{C}$ or $\mathbb{R}$)
$\|\cdot\|$	:	norm on struct
Perturbation level	:	$\rho \geq 0$.

Definition:

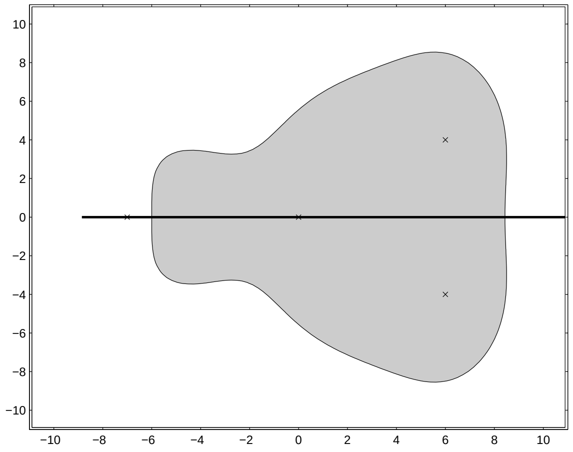
$$\sigma_{\text{struct}}(A,\rho) \quad := \quad \text{Set of eigenvalues of all matrices of the form}$$
$$A + \Delta, \qquad \Delta \in \text{struct}, \ \|\Delta\| \leq \rho.$$

Example:

$\rho = 4.0$



struct $= \mathbb{C}^{n\times n}$



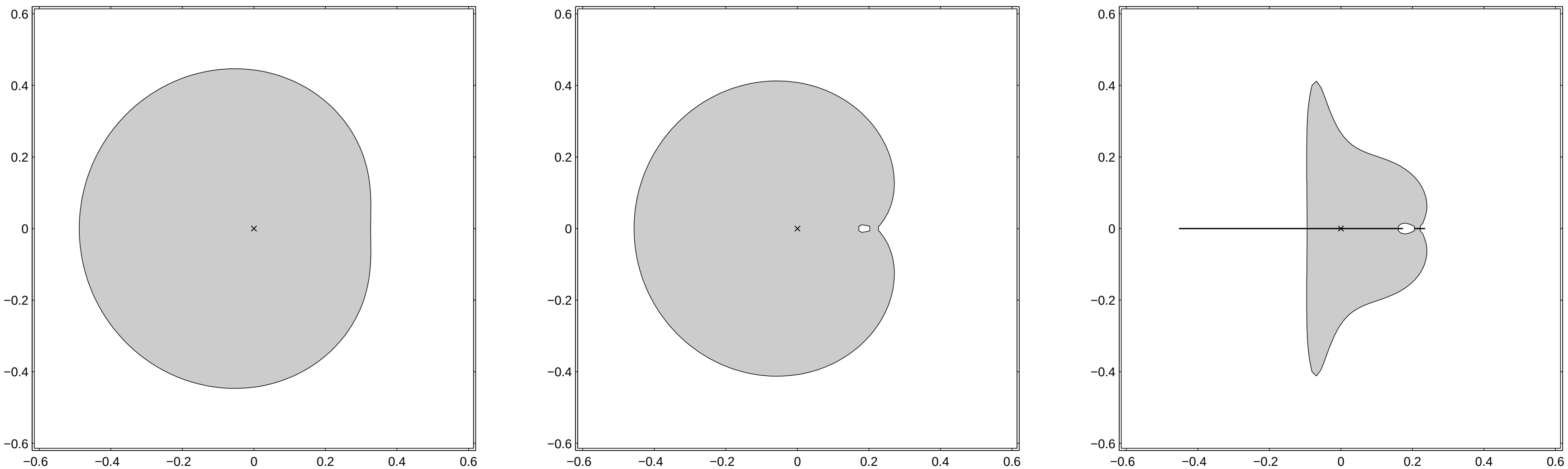
struct $= \mathbb{R}^{n\times n}$

Example: Pseudospectra

$$\sigma_{\mathbb{F}}(A,\rho) = \{\, s \in \mathbb{C} \;\mid\; s \text{ is eigenvalue of } A + \Delta \text{ for some } \Delta \in \mathbb{F}^{n\times n} \text{ with } \|\Delta\| \le \rho \,\},$$

where

$$A = \begin{bmatrix} 0 & 1 & -6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \rho = 0.025.$$

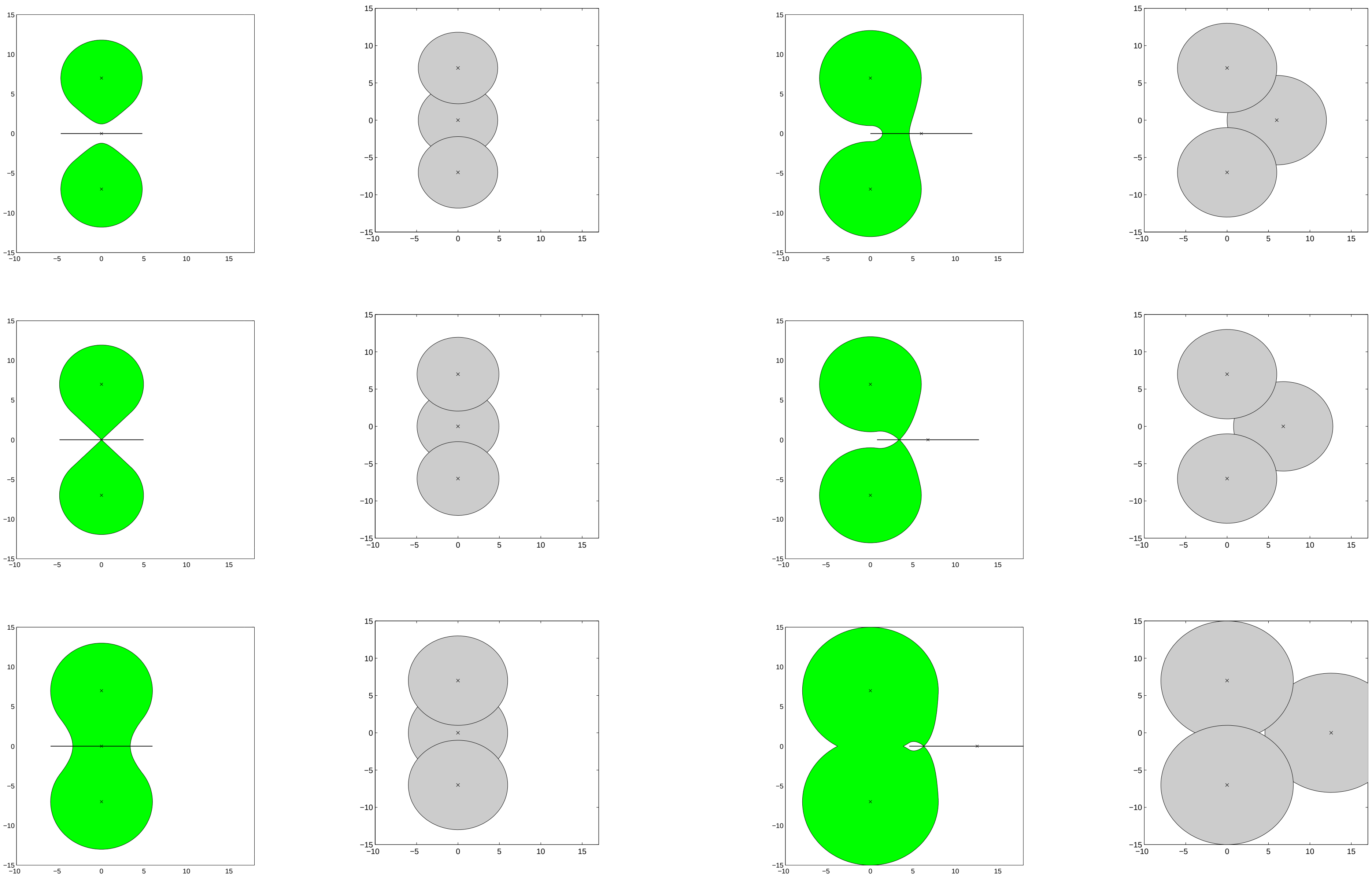


$\mathbb{F} = \mathbb{C} \quad \|\cdot\| = \|\cdot\|_1$ $\qquad$ $\mathbb{F} = \mathbb{C} \quad \|\cdot\| = \|\cdot\|_2$ $\qquad$ $\mathbb{F} = \mathbb{R} \quad \|\cdot\| = \|\cdot\|_2$

Pseudospectra $\sigma_{\mathbb{F}}(A, \rho)$ **of a real normal** 3×3 **matrix.**

Underlying norm is the spectral norm.

Real perturbations in green, complex perturbations in grey.



Pseudospectra and the structured distance to singularity

Structured distance of $M \in \mathbb{C}^{n\times n}$ to the set of singular matrices:

$$d_{\text{struct}}(M) := \inf\{\|\Delta\|;\ \ \Delta \in \text{struct},\ M - \Delta \text{ is singular}\}.$$

For any $s \in \mathbb{C}$, $A \in \mathbb{C}^{n\times n}$,

$$\begin{aligned} d_{\text{struct}}(sI - A) &= \inf\{\,\|\Delta\|;\ \ \Delta \in \text{struct},\ \underbrace{(sI - A) - \Delta}_{sI-(A+\Delta)} \text{ is singular}\,\} \\ &= \inf\{\,\|\Delta\|;\ \ \Delta \in \text{struct},\ s \text{ is eigenvalue of } A + \Delta\,\}. \end{aligned}$$

We have

$$\begin{aligned} s \in \sigma_{\text{struct}}(A,\rho) \quad &\Leftrightarrow \quad s \text{ is an eigenvalue of } A + \Delta \\ &\qquad \text{for some } \Delta \in \text{struct},\ \|\Delta\| \le \rho \\ &\Leftrightarrow \quad d_{\text{struct}}(sI - A) \le \rho. \end{aligned}$$

Thus

$$\boxed{\sigma_{\text{struct}}(A,\rho) \;:=\; \{\, s \in \mathbb{C};\ \ d_{\text{struct}}(sI - A) \le \rho\,\}.}$$

Formula for structured pseudospectra:

$$\sigma_{\text{struct}}(A,\rho) \quad := \quad \{\, s \in \mathbb{C};\;\; d_{\text{struct}}(sI - A) \le \rho \,\}.$$

Hence: pseudospectra can be plottet if $d_{\text{struct}}(M)$ can be calculated.

Well known:

$$d_{\mathbb{C}^{n\times n}}(M) \;=\; \sigma_{\min}(M)$$

$$d_{\mathbb{R}^{n\times n}}(M) \;=\; \sup_{\gamma\in(0,1]} \sigma_{2n-1}\left(\begin{bmatrix} \Re M & -\gamma\,\Im M \\ \gamma^{-1}\,\Im M & \Re M \end{bmatrix}\right)$$

(Bernhardson, Davidson, Doyle, Qui, Rantzer, Young, 1993)

$$d_{Ham}(M) \;=\; \sqrt{\max_{t\in\mathbb{R}} \lambda_{\min}(M^*M + t\,i\,(MJ + JM^*))}, \qquad \text{(K. 2007)}$$

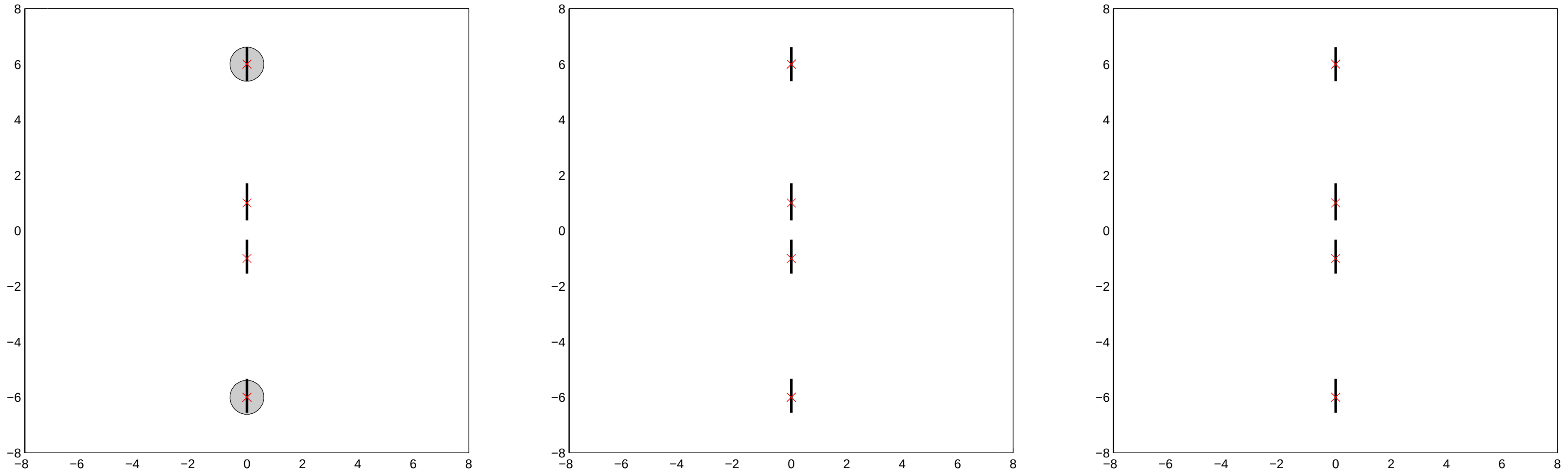
where

$$Ham = \left\{ \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix};\;\; B = B^*, C = C^* \right\}, \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Example: Pseudospectra $\sigma_{Ham}(H,\rho)$ for the Hamiltonian matrix

$$H = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}. \qquad \text{Eigenvalues of } H\text{: } \pm i \text{ and } \pm 6i \text{ (double).}$$

$B = \text{diag}(1, 6, -6)$ $\qquad$ $B = \text{diag}(1, 6, 6)$ $\qquad$ $B = \text{diag}(-1, 6, 6)$

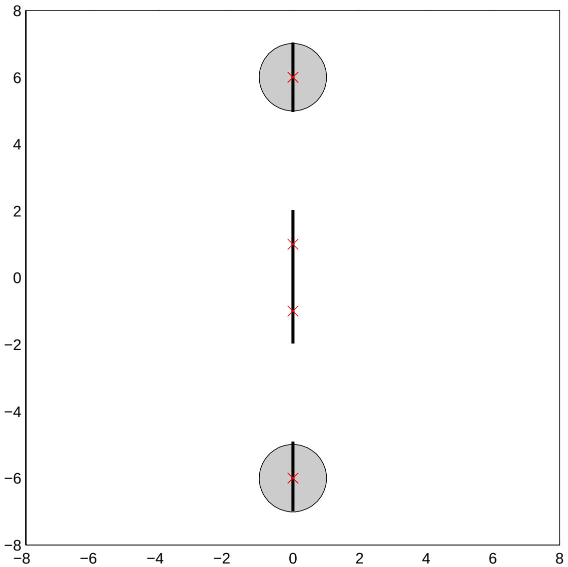


Perturbation level: $\qquad \rho = 0.6$

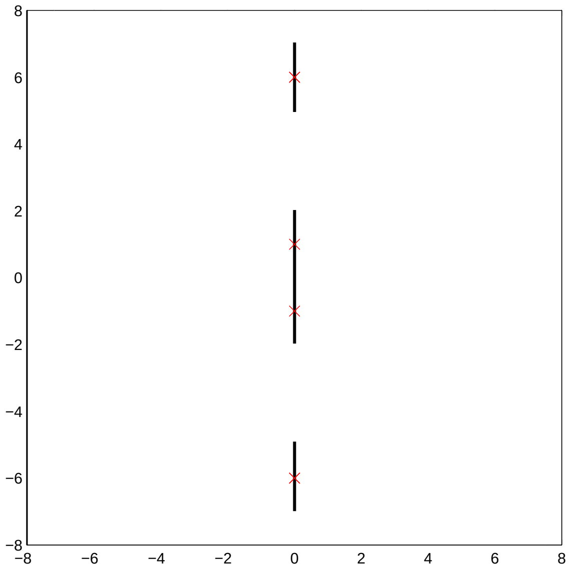
Example: Pseudospectra $\sigma_{Ham}(H,\rho)$ for the Hamiltonian matrix

$$H = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}. \qquad \text{Eigenvalues of } H\text{:} \quad \pm i \text{ and } \pm 6i \text{ (double).}$$

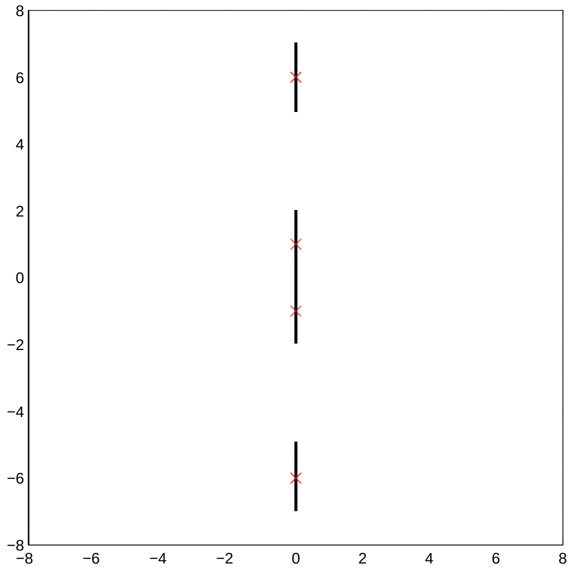
$B = \mathrm{diag}(1, 6, -6)$



$B = \mathrm{diag}(1, 6, 6)$



$B = \mathrm{diag}(-1, 6, 6)$

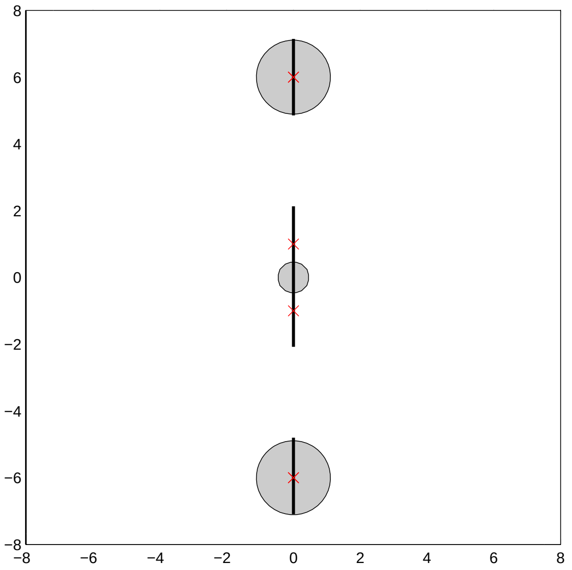


Perturbation level: $\rho = 1.0$

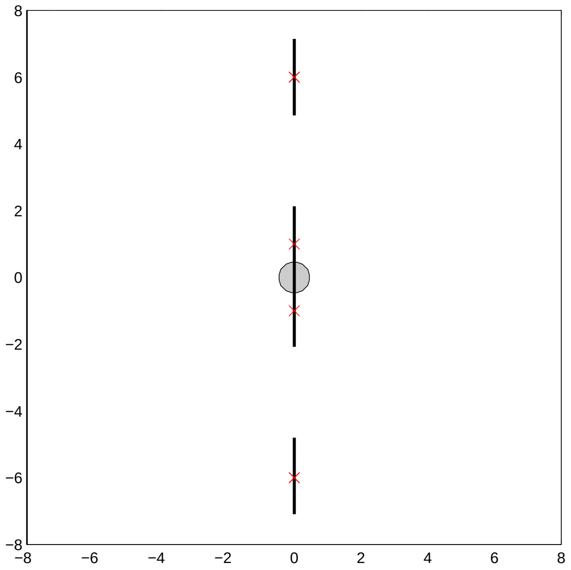
Example: Pseudospectra $\sigma_{Ham}(H,\rho)$ for the Hamiltonian matrix

$$H = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}. \qquad \text{Eigenvalues of } H\text{:} \quad \pm i \text{ and } \pm 6i \text{ (double).}$$

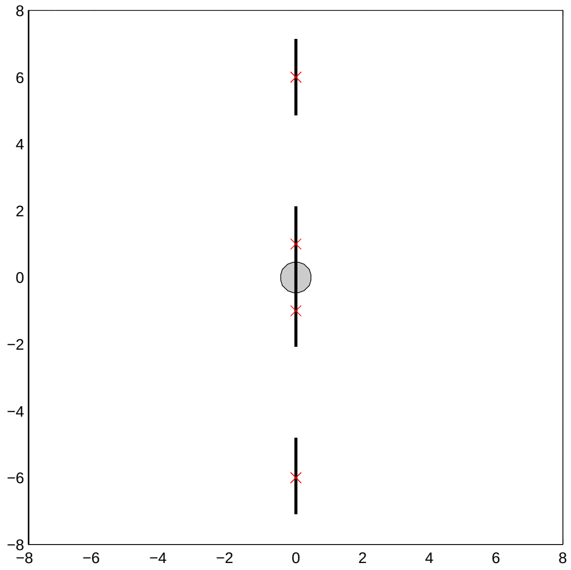
$B = \text{diag}(1, 6, -6)$



$B = \text{diag}(1, 6, 6)$



$B = \text{diag}(-1, 6, 6)$



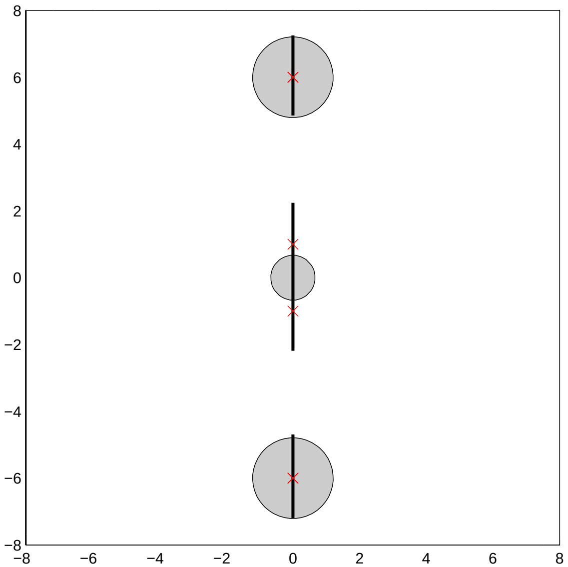
Perturbation level: $\rho = 1.1$

Example: Pseudospectra $\sigma_{Ham}(H, \rho)$ for the Hamiltonian matrix

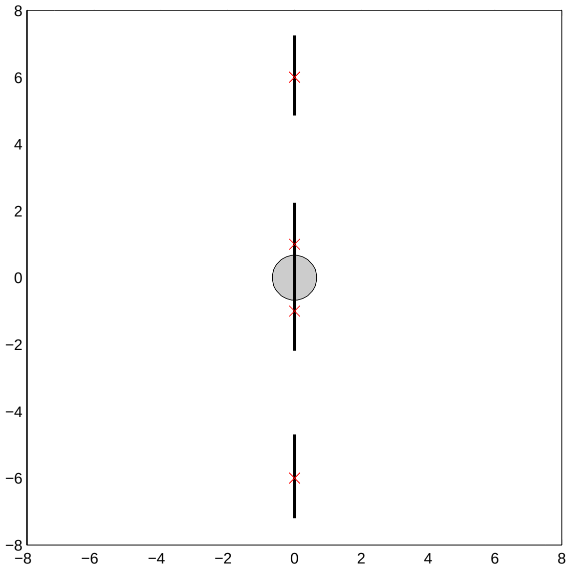
$$H = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}.$$

Eigenvalues of H: $\pm i$ and $\pm 6i$ (double).

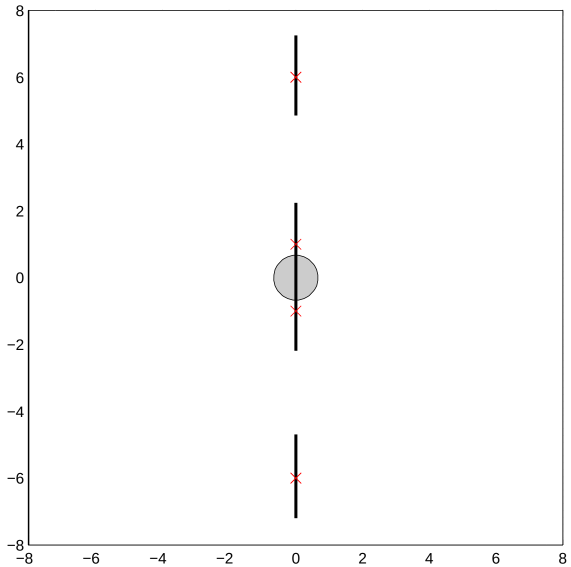
$B = \text{diag}(1, 6, -6)$



$B = \text{diag}(1, 6, 6)$



$B = \text{diag}(-1, 6, 6)$



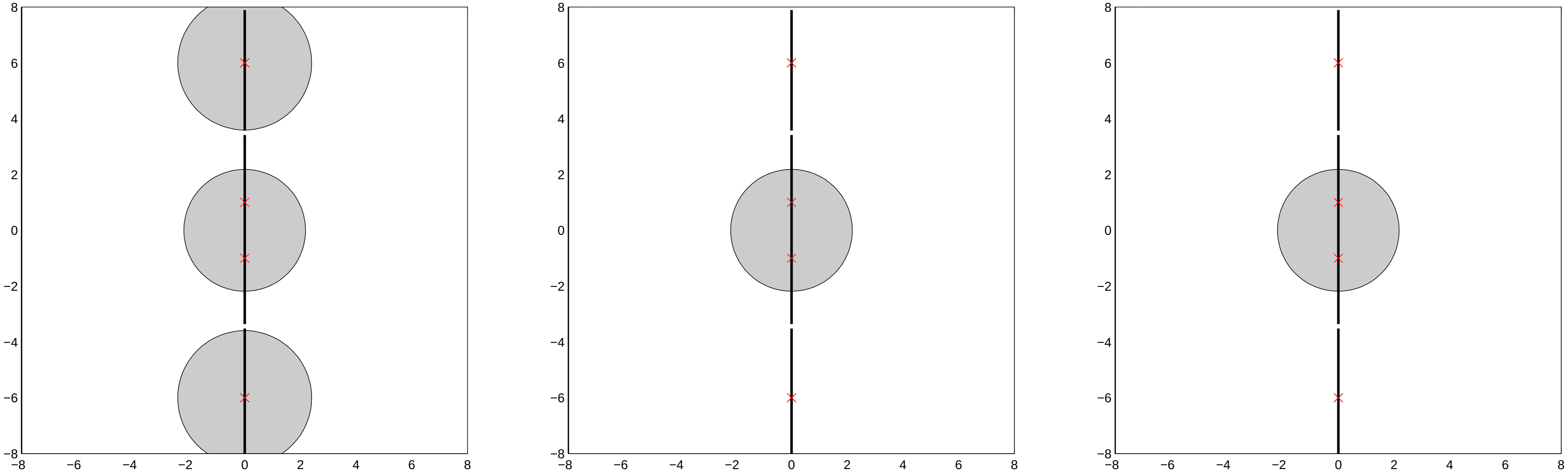
Perturbation level: $\rho = 1.2$

Example: Pseudospectra $\sigma_{Ham}(H,\rho)$ for the Hamiltonian matrix

$$H = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}.$$

Eigenvalues of H: $\pm i$ and $\pm 6i$ (double).

$B = \text{diag}(1,6,-6)$ $\quad$ $B = \text{diag}(1,6,6)$ $\quad$ $B = \text{diag}(-1,6,6)$



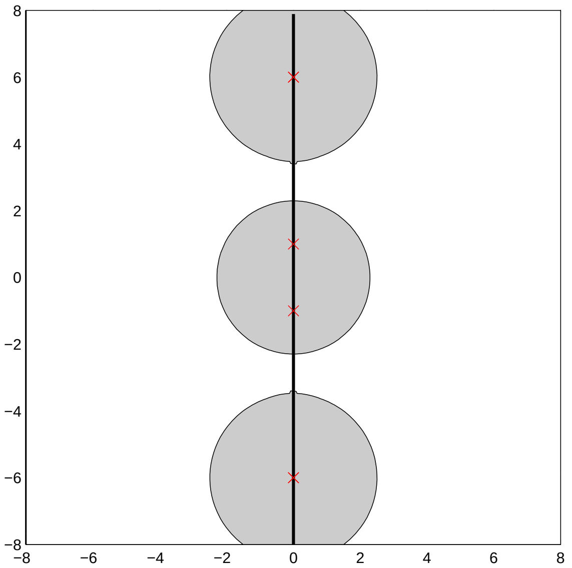
Perturbation level: $\rho = 2.4$

Example: Pseudospectra $\sigma_{Ham}(H, \rho)$ for the Hamiltonian matrix

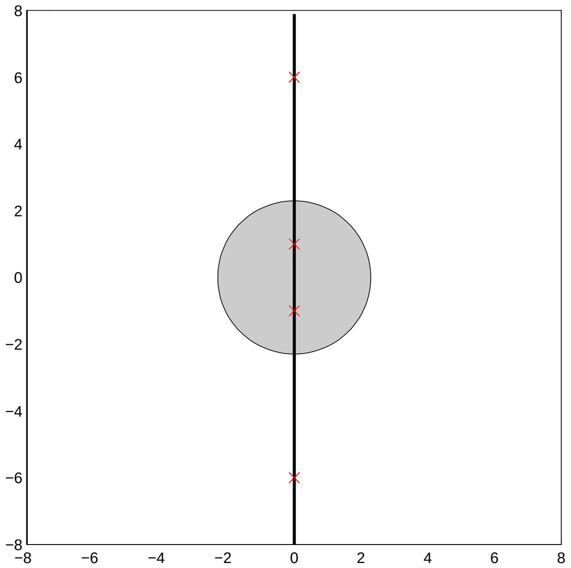
$$H = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}.$$

Eigenvalues of H: $\pm i$ and $\pm 6i$ (double).

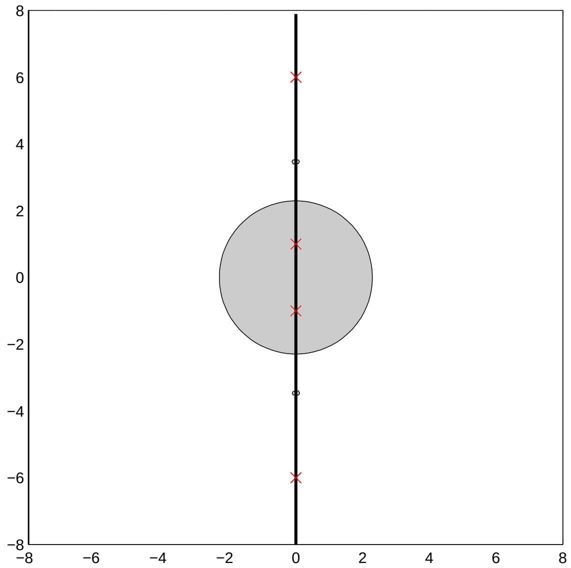
$B = \text{diag}(1, 6, -6)$



$B = \text{diag}(1, 6, 6)$



$B = \text{diag}(-1, 6, 6)$

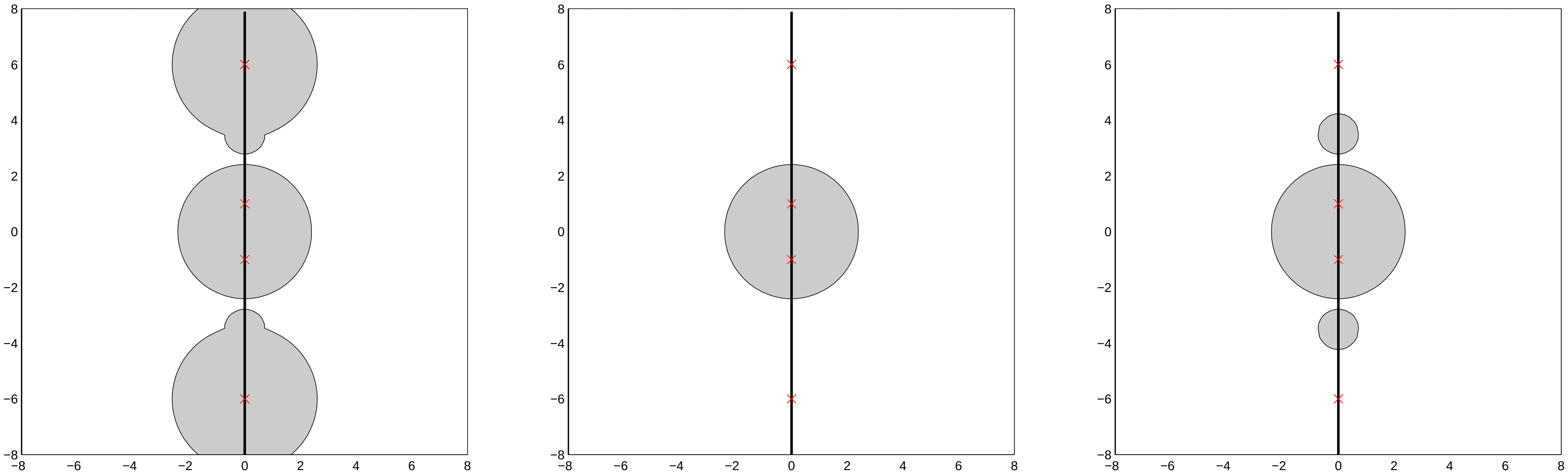


Perturbation level: $\rho = 2.5$

Example: Pseudospectra $\sigma_{Ham}(H,\rho)$ for the Hamiltonian matrix

$$H = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}. \qquad \text{Eigenvalues of } H\text{:} \quad \pm i \text{ and } \pm 6i \text{ (double).}$$

$B = \text{diag}(1, 6, -6)$ $\qquad$ $B = \text{diag}(1, 6, 6)$ $\qquad$ $B = \text{diag}(-1, 6, 6)$



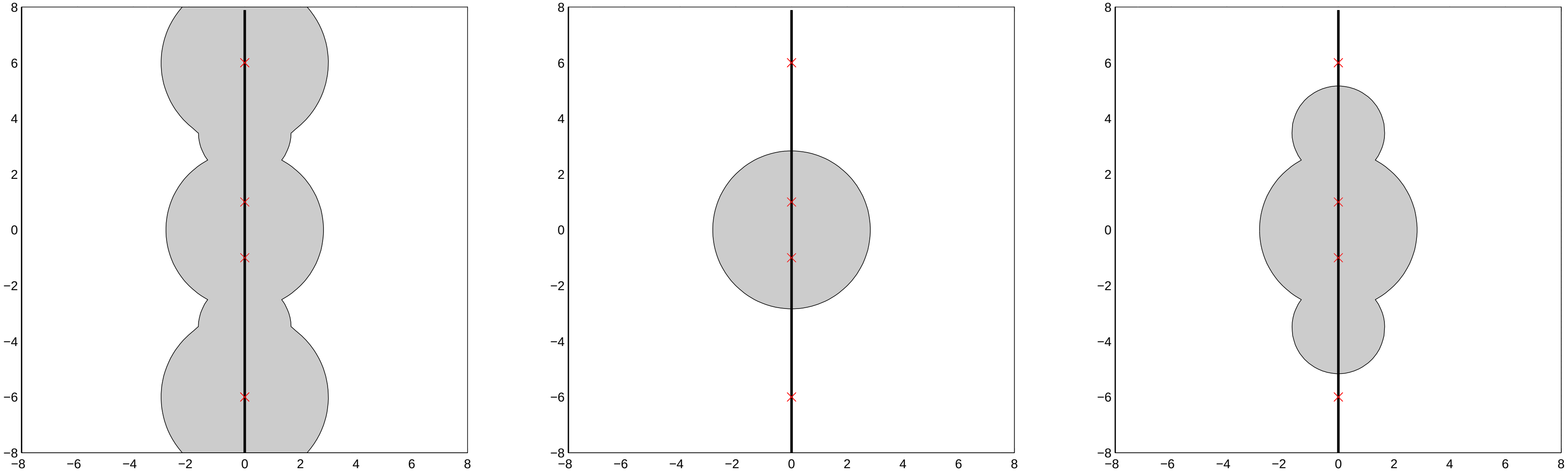
Perturbation level: $\qquad \rho = 2.6$

Example: Pseudospectra $\sigma_{Ham}(H, \rho)$ for the Hamiltonian matrix

$$H = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}.$$

Eigenvalues of H: $\pm i$ and $\pm 6i$ (double).

$B = \text{diag}(1, 6, -6)$ $\qquad$ $B = \text{diag}(1, 6, 6)$ $\qquad$ $B = \text{diag}(-1, 6, 6)$

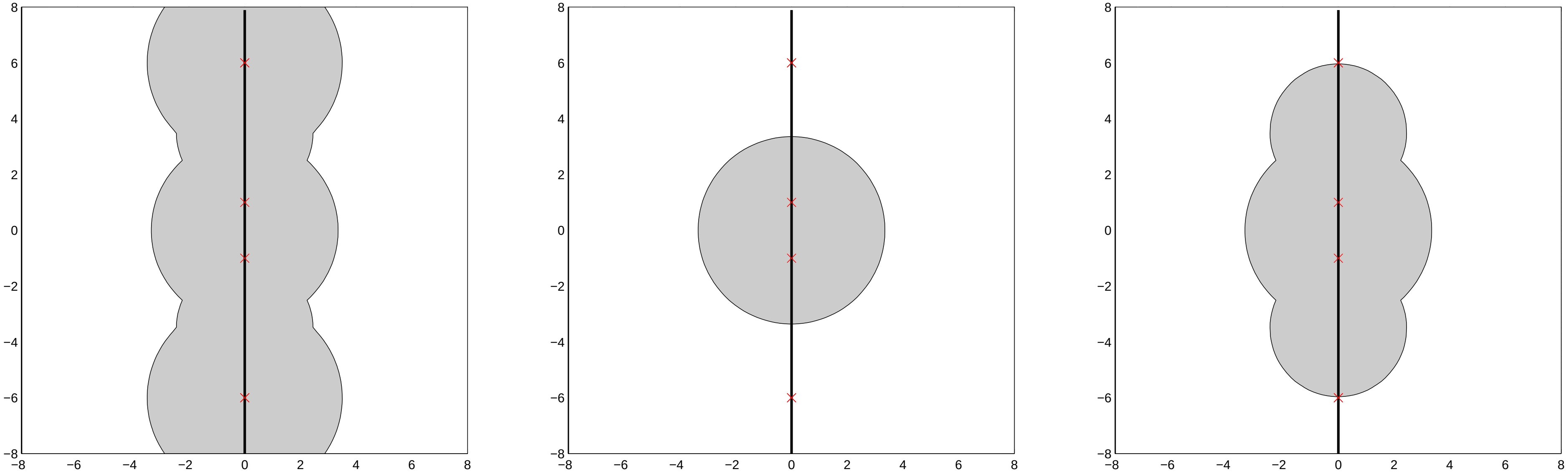


Perturbation level: $\rho = 3.0$

Example: Pseudospectra $\sigma_{Ham}(H,\rho)$ for the Hamiltonian matrix

$$H = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}. \qquad \text{Eigenvalues of } H\text{:} \quad \pm i \text{ and } \pm 6i \text{ (double).}$$

$B = \text{diag}(1, 6, -6)$ $\qquad$ $B = \text{diag}(1, 6, 6)$ $\qquad$ $B = \text{diag}(-1, 6, 6)$



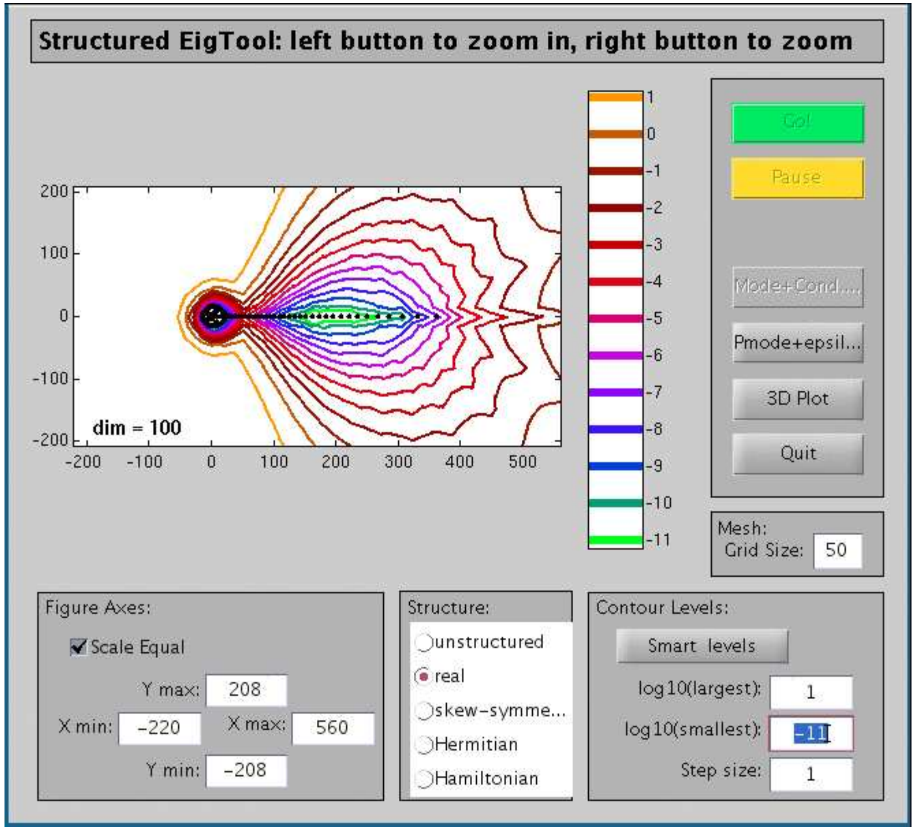
Perturbation level: $\qquad \rho = 3.5$

Algorithms for computing structured pseudospectra are given in

M. Karow, E. Kokiopoulou, and D. Kressner.

On the computation of structured singular values and pseudospectra.

Technical report 2009-26, ETH Zurich, August 2009.



MATLAB-code 'Eigtool' (by T. Wright, N. Trefethen, M. Embree, 2002) has been modified to compute structured pseudospectra.

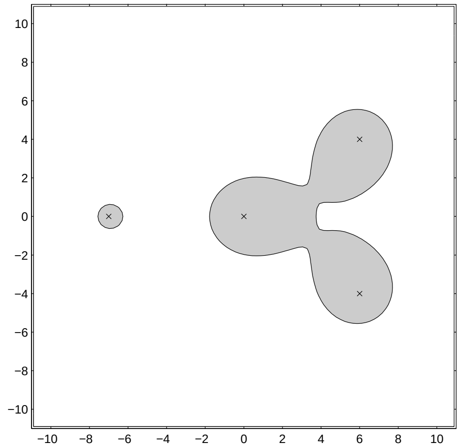
Small structured perturbations of a simple eigenvalue

Papers:

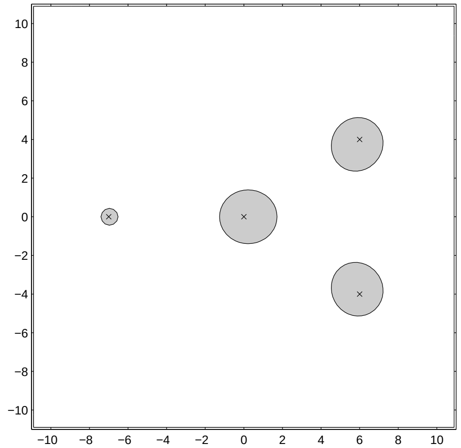
M. Karow, D. Kressner, F. Tisseur.
Structured Eigenvalue Condition Numbers.
SIAM J. Matrix Anal. Appl. 28(4):1052-1068. (2006)

Michael Karow.
Pseudospectra and the condition of a nonderogatory eigenvalue.
SIAM. J. Matrix Anal. Appl. 31 (2010)

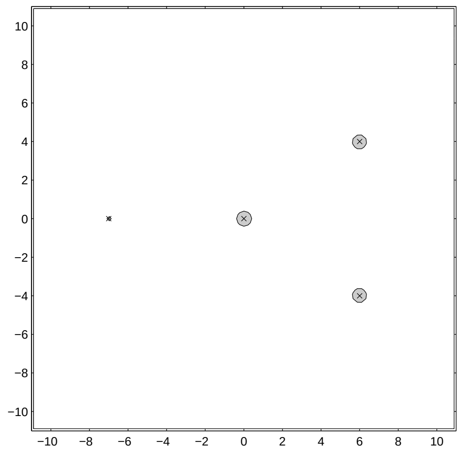
$\rho = .10$

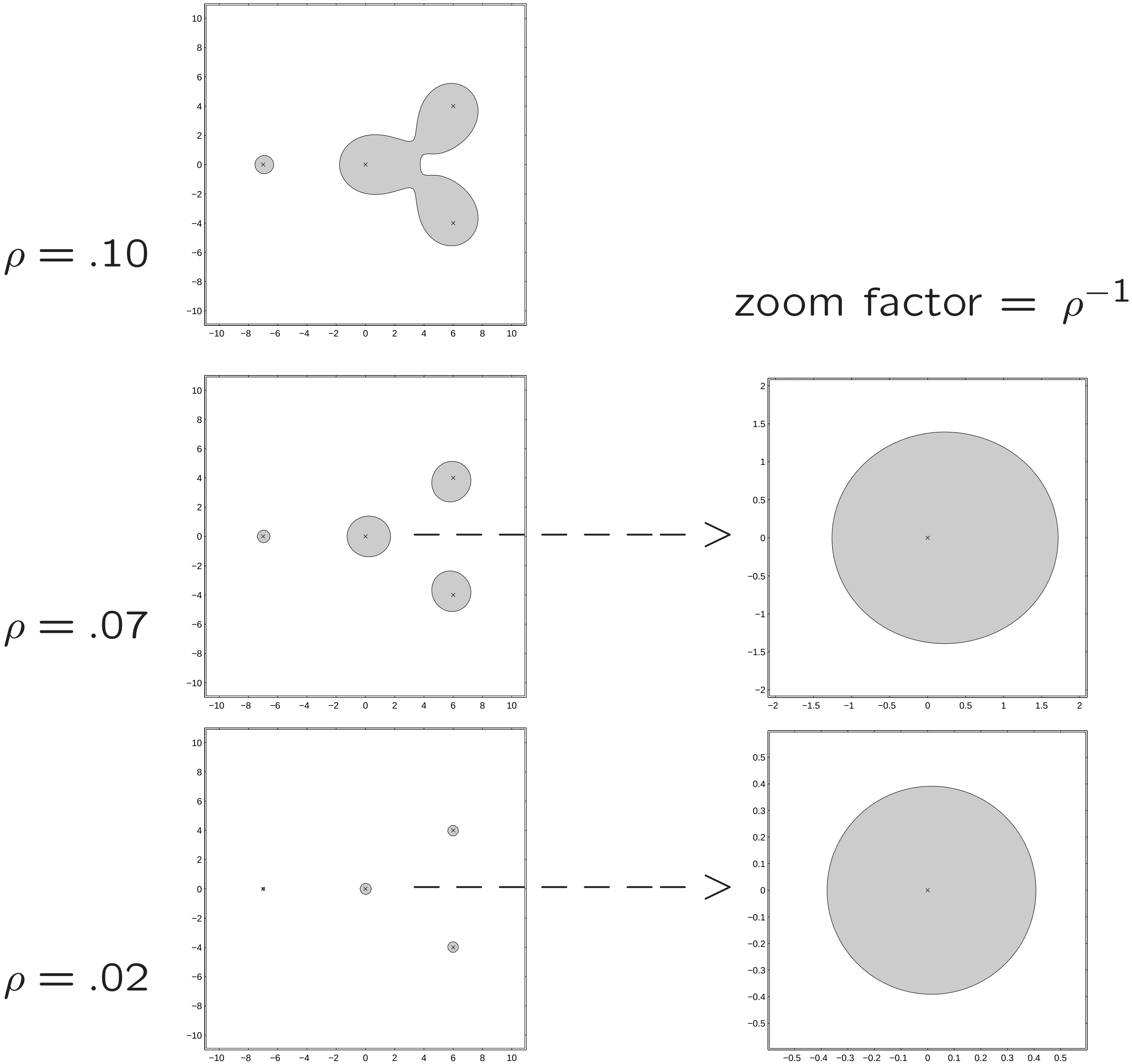


$\rho = .07$

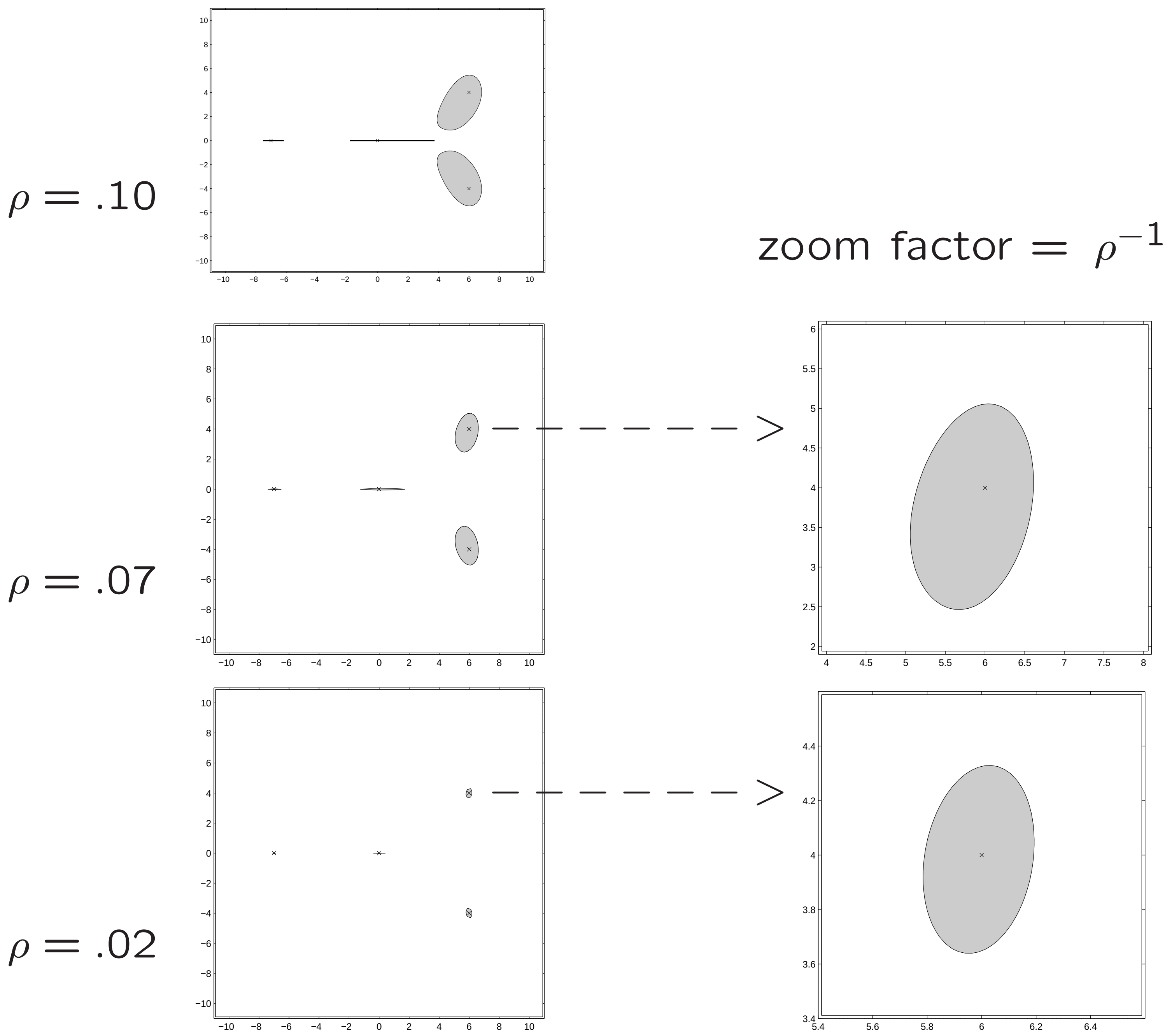


$\rho = .02$





$\rho = .10$
zoom factor $= \rho^{-1}$
$\rho = .07$
$\rho = .02$


ρ = .10
ρ = .07
ρ = .02
zoom factor = ρ^−1

Theorem: Let

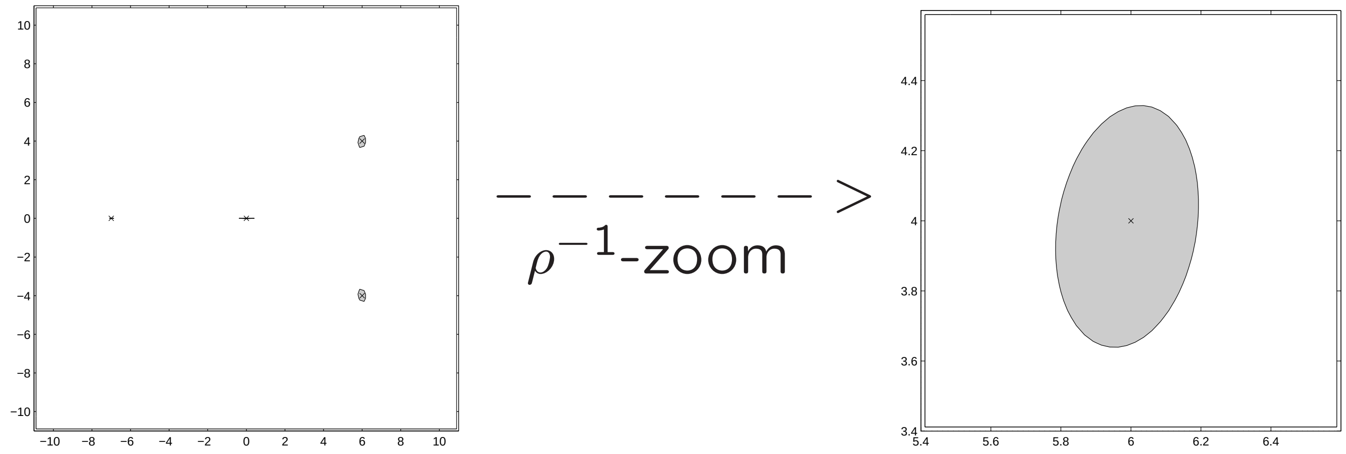
- $\lambda \in \mathbb{C}$ be a simple eigenvalue of $A \in \mathbb{C}^n$,
- (x, y) be a pair of right and left eigenvectors such that $y^*x = 1$,
- $K_{\text{struct}}(x, y) := \{\, y^*\Delta x; \;\; \Delta \in \text{struct}, \;\; \|\Delta\| \leq 1 \,\}$
- $\mathcal{C}_{\text{struct}}(\lambda, \rho) =$ the connected component of the structured pseudospectrum $\sigma_{\text{struct}}(A, \rho)$ that contains λ.

Then with respect to the Hausdorff metric,

$$\lim_{\rho \to 0} \frac{\mathcal{C}_{\text{struct}}(\lambda, \rho) - \lambda}{\rho} = K_{\text{struct}}(x, y),$$

Roughly:

$$\mathcal{C}_{\text{struct}}(\lambda, \rho) \approx \lambda + \rho \, K_{\text{struct}}(x, y) \qquad \text{for small } \rho$$



We have

$$\mathcal{C}_{\text{struct}}(\lambda, \rho) \;\approx\; \lambda + \rho\, K_{\text{struct}}(x, y) \qquad \text{for small } \rho,$$

where

$$K_{\text{struct}}(x, y) \;:=\; \{\, y^* \Delta x;\;\; \Delta \in \text{struct},\;\; \|\Delta\| \le 1 \,\}$$

Reason: Simple eigenvalues are differentiable:

$$\lambda(A + \Delta\, \rho) \;=\; \lambda + \; y^* \Delta x\, \rho + \mathcal{O}(\rho^2),$$

The **structured condition number** of λ is

$$\text{cond}_{\text{struct}}(A, \lambda) \;=\; \max_{\substack{\Delta \in \text{struct} \\ \|\Delta\| \le 1}} |y^* \Delta x| \;=\; \max\{\, |z|;\;\; z \in K_{\text{struct}}(x, y)\, \}.$$

Problem: Determine the following subsets of $\mathbb{C}$:

$$K_{\text{struct}}(x,y) = \{\, y^* \Delta x;\ \ \Delta \in \text{struct},\ \ \|\Delta\| \le 1\,\},$$

where $x, y \in \mathbb{C}^n$, $\|\cdot\|$ norm on $\mathbb{C}^{n\times n}$ and

$$\text{struct} \subseteq \mathbb{C}^{n\times n} \text{ subspace over } \mathbb{R}$$

for instance,

$$\begin{array}{rcl} \mathbb{F}^{n\times n} & & \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}, \\ \text{Herm} & = & \{\, \Delta \in \mathbb{C}^{n\times n};\quad \Delta^* = \Delta \,\}, \\ \text{Sym}_{\mathbb{F}} & = & \{\, \Delta \in \mathbb{F}^{n\times n};\quad \Delta^\top = \Delta \,\}, \\ \text{Skew}_{\mathbb{F}} & = & \{\, \Delta \in \mathbb{F}^{n\times n};\quad \Delta^\top = -\Delta \,\}. \end{array}$$

Problem: Computation of $K_{\text{struct}}(x, y)$.

Recall: $K_{\text{struct}}(x, y) = \{ y^* \Delta x; \;\; \Delta \in \text{struct}, \;\; \|\Delta\| \leq 1 \},$

Key Observation:

If struct $\subseteq \mathbb{C}^{n \times n}$ is convex then $K_{\text{struct}}(x, y) \subset \mathbb{C}$ is convex.

Fact:

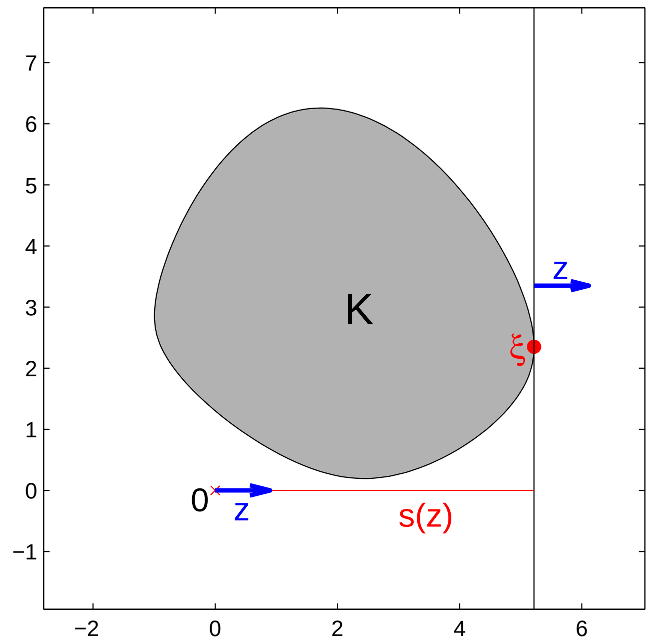
A convex sets can be calculated via its support function ($\rightarrow$ next slide)

Compact convex sets and support functions

The support function of a compact convex set $K \subset \mathbb{C}$ is defined as

$$s(z) = \max_{\xi \in K} z^\top \xi = \max_{\xi \in K} \Re(\bar{z}\,\xi), \qquad \begin{array}{l} z = z_1 + i z_2 \in \mathbb{C} \\ \xi = \xi_1 + i\xi_2 \in \mathbb{C} \end{array}$$

If $|z| = 1$ and ξ is a maximizer then ξ is a boundary point of K.



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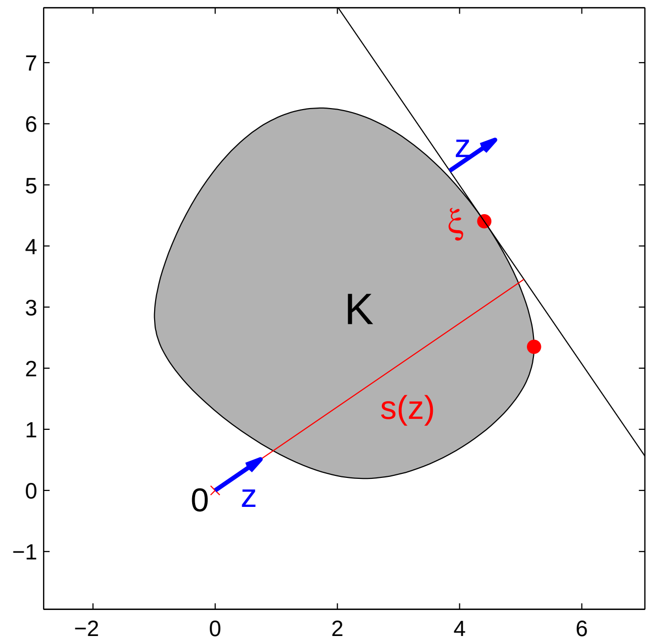

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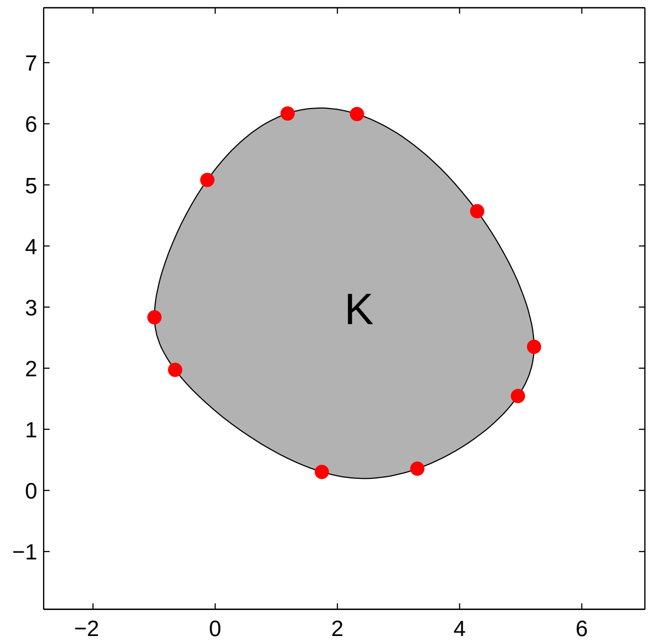

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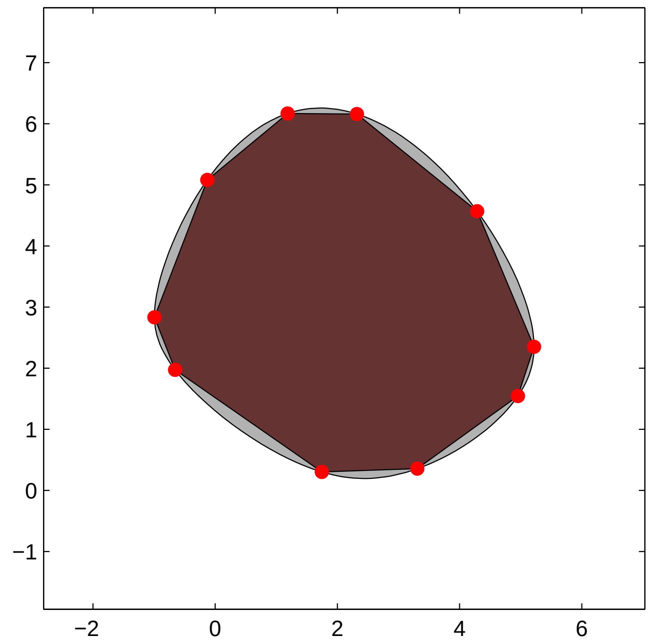


Recall: $K_{\text{struct}}(x,y) = \{\, y^*\Delta x;\;\; \Delta \in \text{struct},\;\; \|\Delta\| \le 1\,\}$,

Theorem: The support function of $K_{\text{struct}}(x,y)$ is

$$
\begin{aligned}
s_{\text{struct}}(z) &= \max_{\xi \in K_{\text{struct}}(x,y)} \Re(\bar{z}\,\xi) \\
&= \max_{\substack{\Delta \in \text{struct} \\ \|\Delta\| \le 1}} \langle z\, yx^*, \Delta\rangle, \qquad\qquad \langle A, B\rangle := \Re\,\text{tr}(A^*B). \\
&= \|\mathcal{P}_{\text{struct}}(z\, yx^*)\|',
\end{aligned}
$$

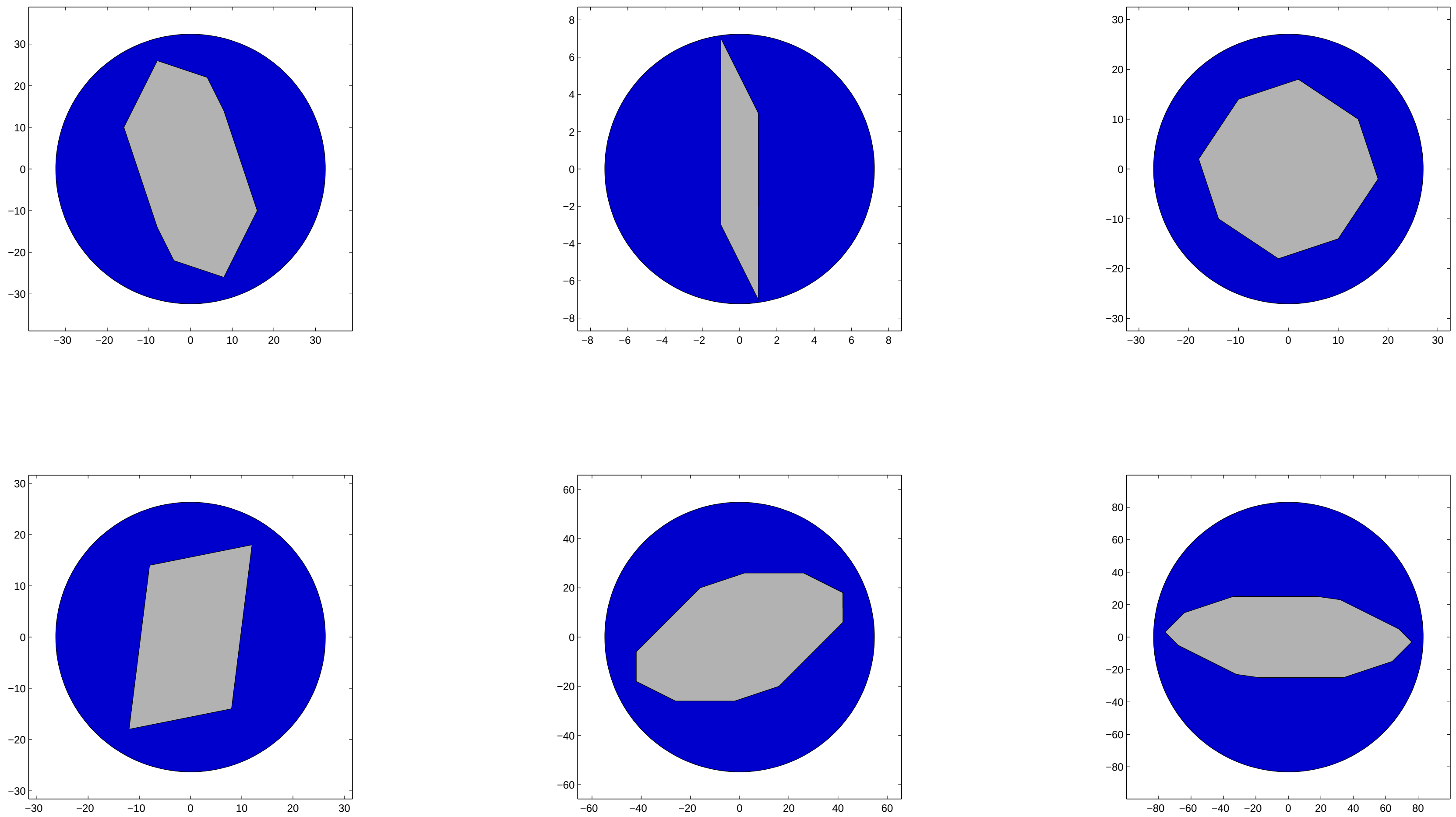
where $\mathcal{P}_{\text{struct}}$ denotes the orthogonal projection onto struct and $\|\cdot\|'$ denotes the dual to the norm $\|\cdot\|$.
A maximizer is

$$\Delta = \mathcal{P}_{\text{struct}}(M)/\|\mathcal{P}_{\text{struct}}(M)\|.$$

Example: The sets

$$K_{\mathbb{F}}(x,y) = \{y^*\Delta x; \quad \Delta \in \mathbb{F}^{n\times n},\ \|\Delta\| \le 1\ \}, \qquad \|\Delta\| = \max_{j,k} |\Delta_{jk}|$$

blue: $\mathbb{F} = \mathbb{C}$, grey $\mathbb{F} = \mathbb{R}$.



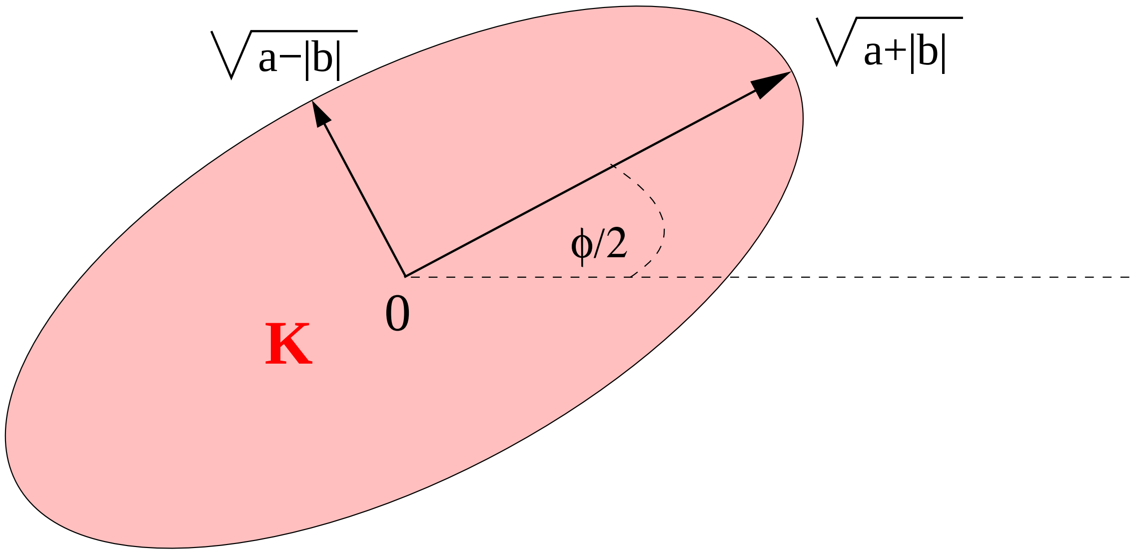
Proposition:

Let $K \subset \mathbb{C}$ be a nonempty compact convex set with support function

$$s_K(z) = \sqrt{a\,|z|^2 + \Re(\bar{b}\,z^2)}, \qquad z, b \in \mathbb{C},\ b = |b|\,e^{i\phi},\ a \geq |b|.$$

Then K is an ellipse. Specifically, $K = e^{i\phi/2}E$, where

$$E = \left\{ \sqrt{a+|b|}\,x + \sqrt{a-|b|}\,y\,i; \quad x, y \in \mathbb{R},\ x^2 + y^2 \leq 1 \right\}.$$



Theorem: If the underlying norm is the Frobenius norm or the spectral norm then $K_{\text{struct}}(x,y)$ is an ellipse with constants a, b and struct specified in the following tables.

Table for the Frobenius norm:

struct	a	b
$\mathbb{C}^{n\times n}$	$\|x\|^2\|y\|^2$	0
$\mathbb{R}^{n\times n}$	$\frac{1}{2}\|x\|^2\|y\|^2$	$\frac{1}{2}(x^\top x)\,(\overline{y^\top y})$
Herm	$\frac{1}{2}\|x\|^2\|y\|^2$	$\frac{1}{2}(y^*x)^2$
$\text{Sym}_{\mathbb{C}}$	$\frac{1}{2}(\|x\|^2\|y\|^2+\|x^\top y\|^2)$	0
$\text{Skew}_{\mathbb{C}}$	$\frac{1}{2}(\|x\|^2\|y\|^2-\|x^\top y\|^2)$	0
$\text{Sym}_{\mathbb{R}}$	$\frac{1}{4}(\|x\|^2\|y\|^2+\|x^\top y\|^2)$	$\frac{1}{4}(\,(x^\top x)\,\overline{(y^\top y)}+(y^*x)^2)$
$\text{Skew}_{\mathbb{R}}$	$\frac{1}{4}(\|x\|^2\|y\|^2-\|x^\top y\|^2)$	$\frac{1}{4}(\,(x^\top x)\,\overline{(y^\top y)}-(y^*x)^2)$

Table for the spectral norm:

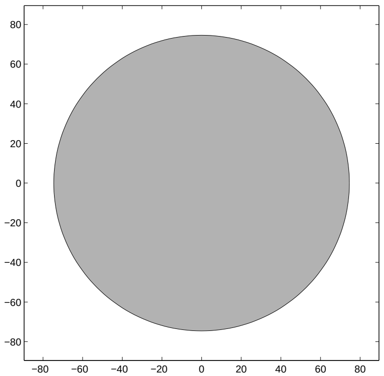
struct	a	b
$\mathbb{C}^{n\times n}$	$\|x\|^2\|y\|^2$	0
$\mathbb{R}^{n\times n}$	$\frac{1}{2}\left[\|x\|^2\,\|y\|^2+\sqrt{(\|x\|^4-\|x^\top x\|^2)(\|y\|^4-\|y^\top y\|^2)}\right]$	$\frac{1}{2}\,(x^\top x)(\overline{y^\top y})$
Herm	$\|x\|^2\|y\|^2-\frac{1}{2}\|y^*x\|^2$	$\frac{1}{2}(y^*x)^2$
$\text{Sym}_{\mathbb{C}}$	$\|x\|^2\|y\|^2$	0
$\text{Skew}_{\mathbb{C}}$	$\|x\|^2\|y\|^2-\|x^\top y\|^2$	0

Example: the sets

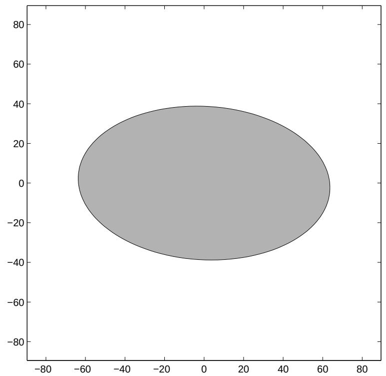
$$K_{\text{struct}}(x,y) \;=\; \{\, y^*\Delta x;\;\; \Delta \in \text{struct},\; \|\Delta\|_F \le 1 \,\},$$

where

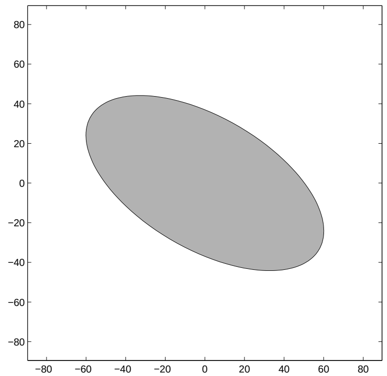
$$x \;=\; [4+3i,\; -1,\; 1+5i,\; -i\,]^\top, \qquad y = [4i,\; 4+3i,\; 4+3i,\; 4+i\,]^\top.$$



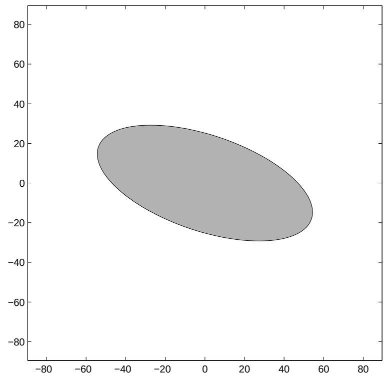
struct $= \mathbb{C}^{n\times n}$



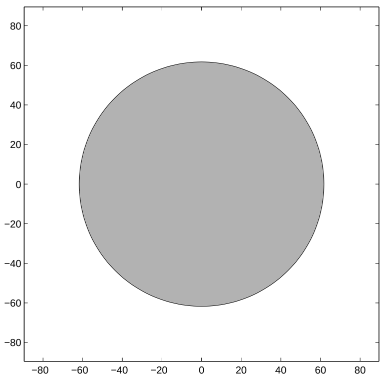
struct $= \mathbb{R}^{n\times n}$



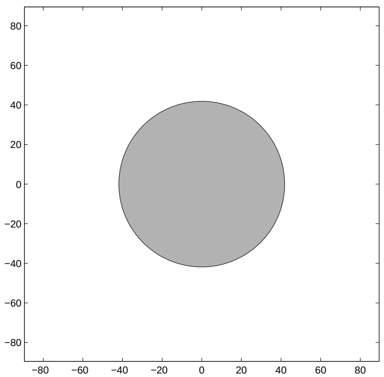
struct $=$ Herm



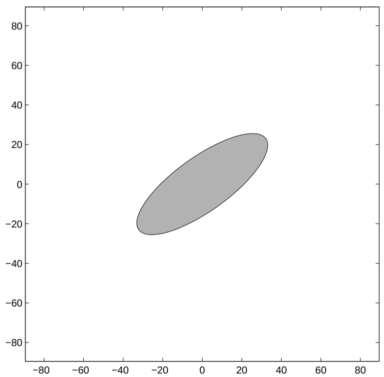
struct $= \text{Sym}_{\mathbb{R}}$



struct $= \text{Sym}_{\mathbb{C}}$



struct $= \text{Skew}_{\mathbb{C}}$



struct $= \text{Skew}_{\mathbb{R}}$

Hamiltonian and skew Hamiltonian perturbations

For the perturbation classes

$$\begin{aligned} \text{Herm} &= \{\, \Delta \in \mathbb{C}^{2n\times 2n}; \quad \Delta^* = \Delta \,\}, \\ \text{Sym}_{\mathbb{F}} &= \{\, \Delta \in \mathbb{F}^{2n\times 2n}; \quad \Delta^\top = \Delta \,\}, \\ \text{Skew}_{\mathbb{F}} &= \{\, \Delta \in \mathbb{F}^{2n\times 2n}; \quad \Delta^\top = -\Delta \,\}, \quad \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}. \end{aligned}$$

one can define the following Hamiltonian classes

$$\text{struct}_{Ham} := \{\, \Delta; \; J\Delta \in \text{struct} \,\}, \quad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

We have

$$K_{\text{struct}_{Ham}}(x, y) = K_{\text{struct}}(x, Jy).$$

Extension of results to nonderogatory eigenvalues

Let λ be a nonderogatory eigenvalue of $A \in \mathbb{C}^{n\times n}$ of algebraic multiplicity m. Let x be a right eigenvector and let $\widehat{y}$ be such that

$$\widehat{y}^*(A-\lambda I)^m = 0, \quad \widehat{y}^* x = 1.$$

Let

$$y^* := \widehat{y}^*(A-\lambda I)^{m-1}.$$

Theorem: We have for the pseudospectra components $\mathcal{C}_{\text{struct}}(\lambda,\rho)$ that

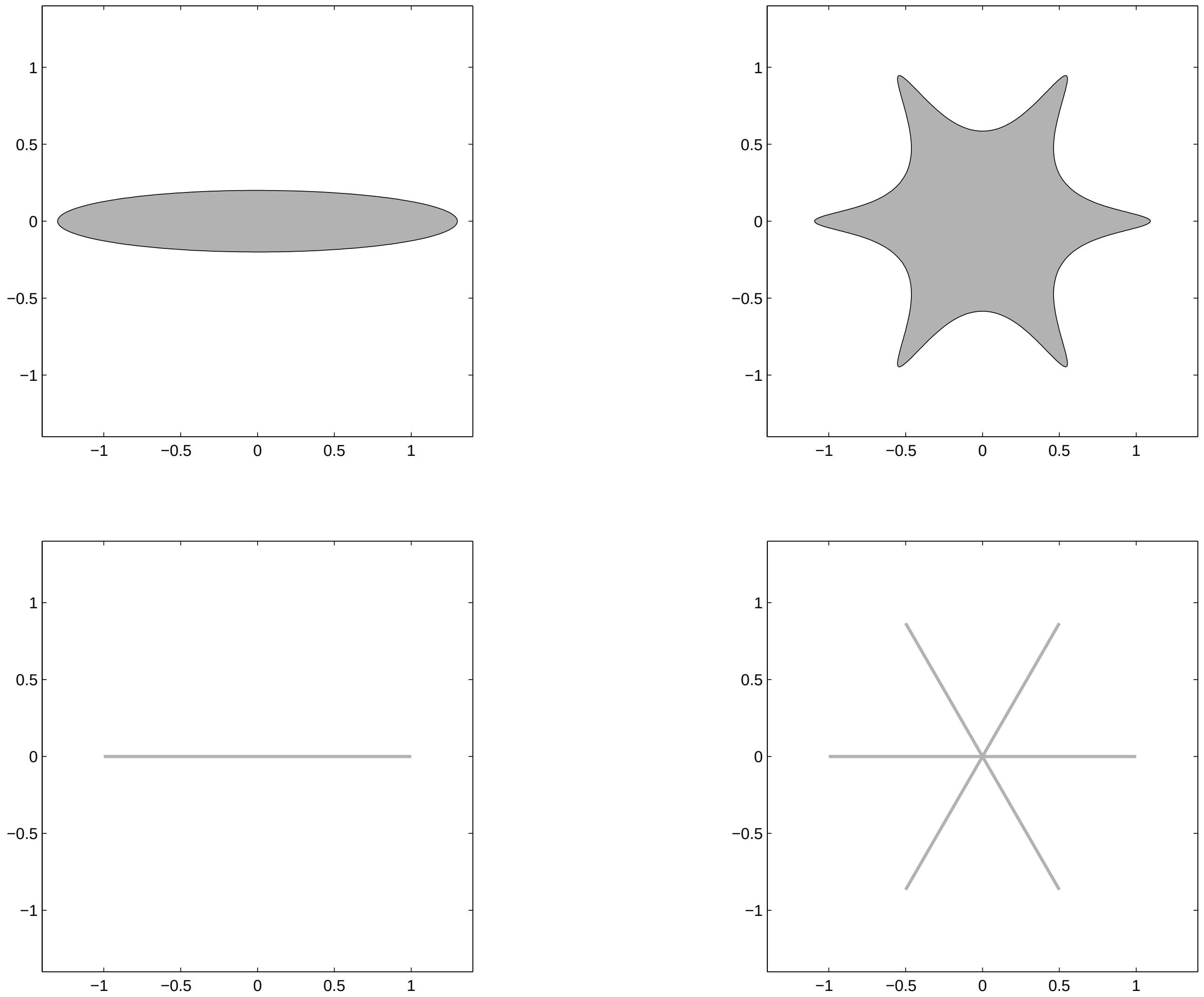
$$\lim_{\rho\to 0} \frac{\mathcal{C}_{\text{struct}}(\lambda,\rho)-\lambda}{\rho^{1/m}} = K_{\text{struct}}{}^{(m)}(x,y),$$

where

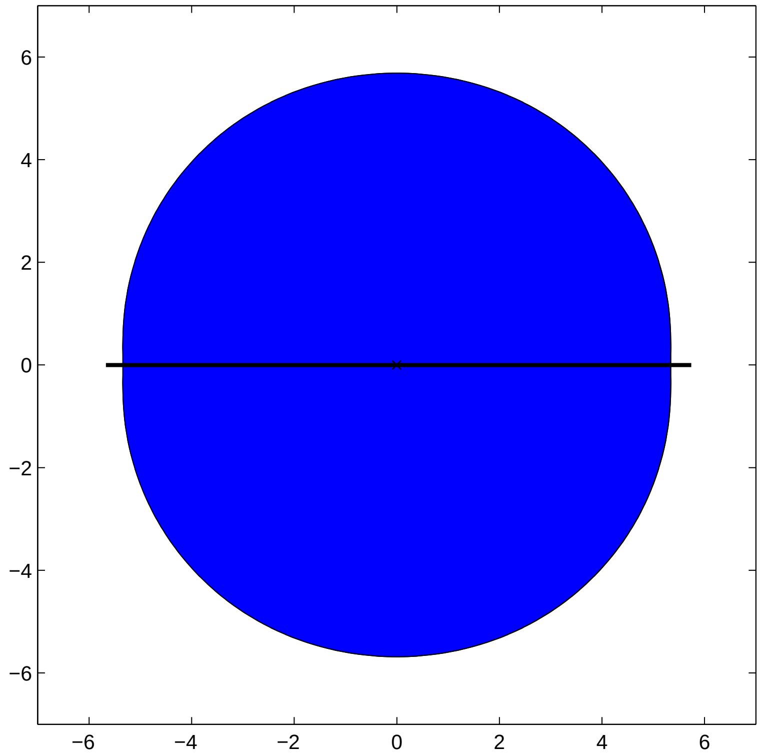
$$K_{\text{struct}}{}^{(m)}(x,y) = \text{ set of } m\text{th roots of } y^*\Delta x, \quad \Delta \in \text{struct}, \quad \|\Delta\| \le 1.$$

Roughly: $\mathcal{C}_{\text{struct}}(\lambda,\rho) \approx \lambda + \rho^{1/m} K_{\text{struct}}{}^{(m)}(x,y)$ for small ρ

$K_{\text{struct}}(x, y)$ $\quad$ ${K_{\text{struct}}}^{(3)}(x, y)$

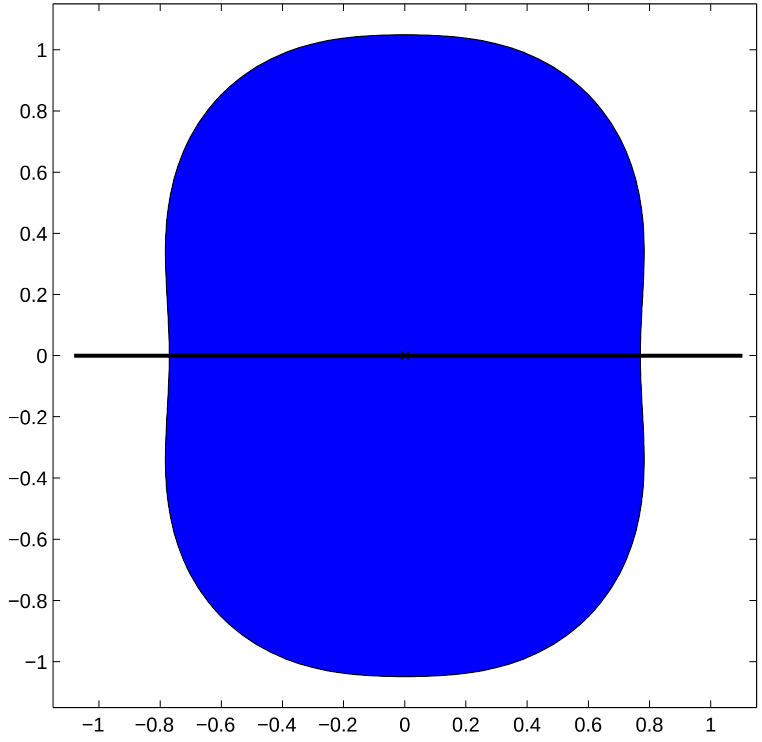


REAL pseudospectra of a REAL 3×3 Jordan block (w.r.t spectral norm)



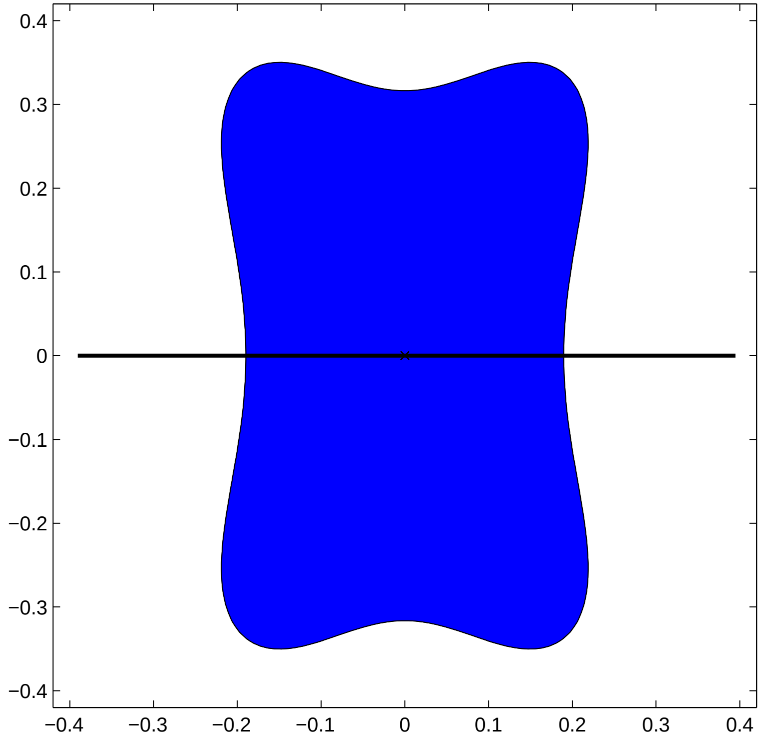
Perturbation level $\rho = 5 * 10^{1}$

REAL pseudospectra of a REAL 3×3 Jordan block (w.r.t spectral norm)



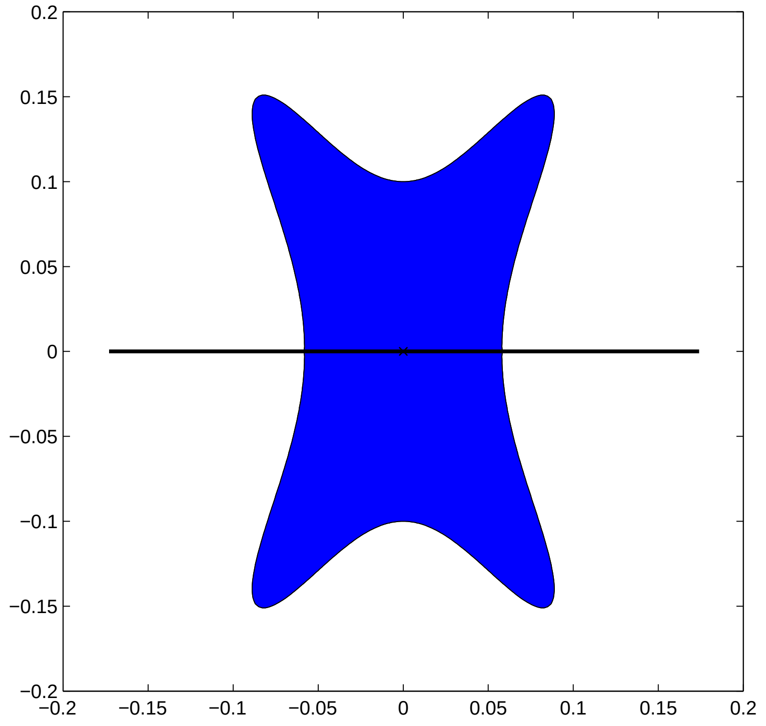
Perturbation level $\rho = 5 * 10^{-1}$

REAL pseudospectra of a REAL 3×3 Jordan block (w.r.t spectral norm)



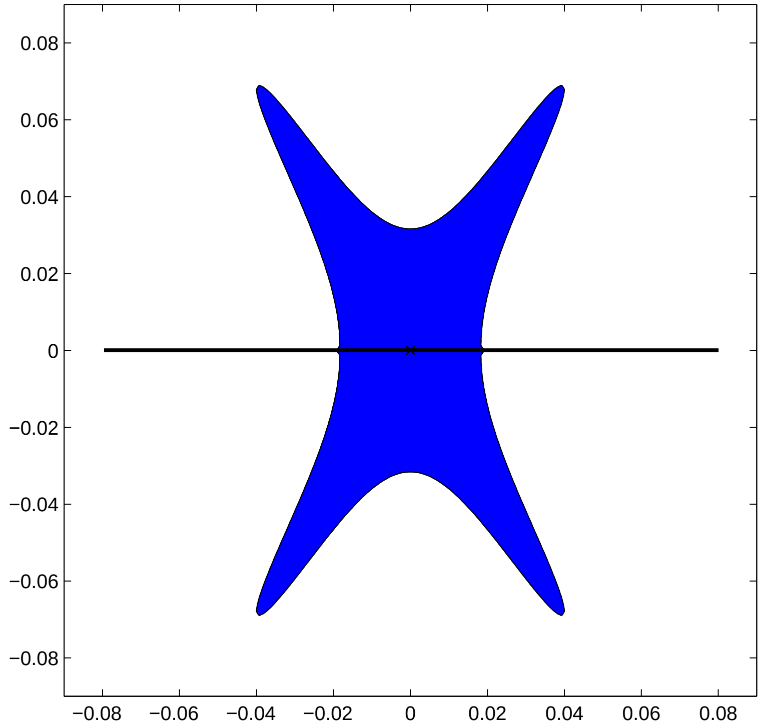
Perturbation level $\rho = 5 * 10^{-2}$

REAL pseudospectra of a REAL 3×3 Jordan block (w.r.t spectral norm)



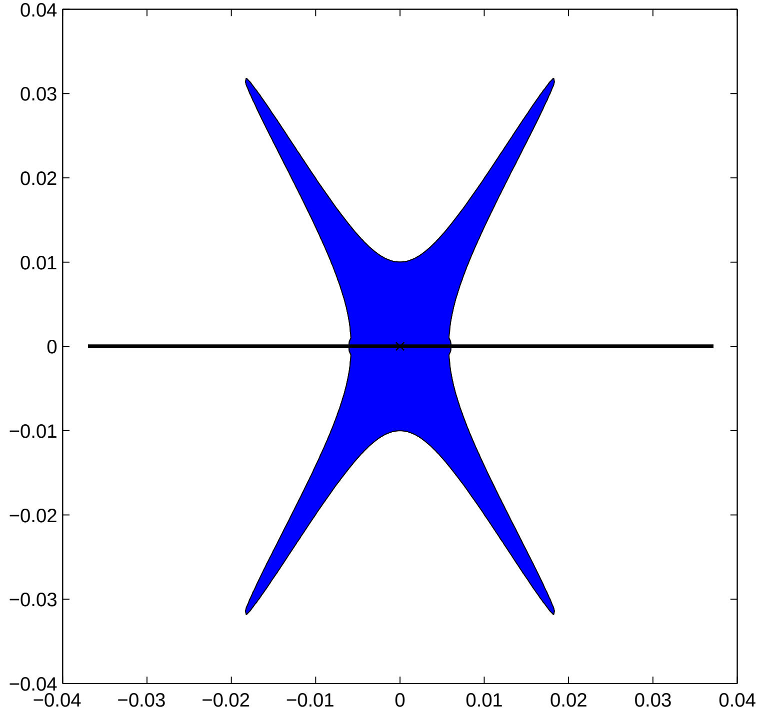
Perturbation level $\rho = 5 * 10^{-3}$

REAL pseudospectra of a REAL 3×3 Jordan block (w.r.t spectral norm)



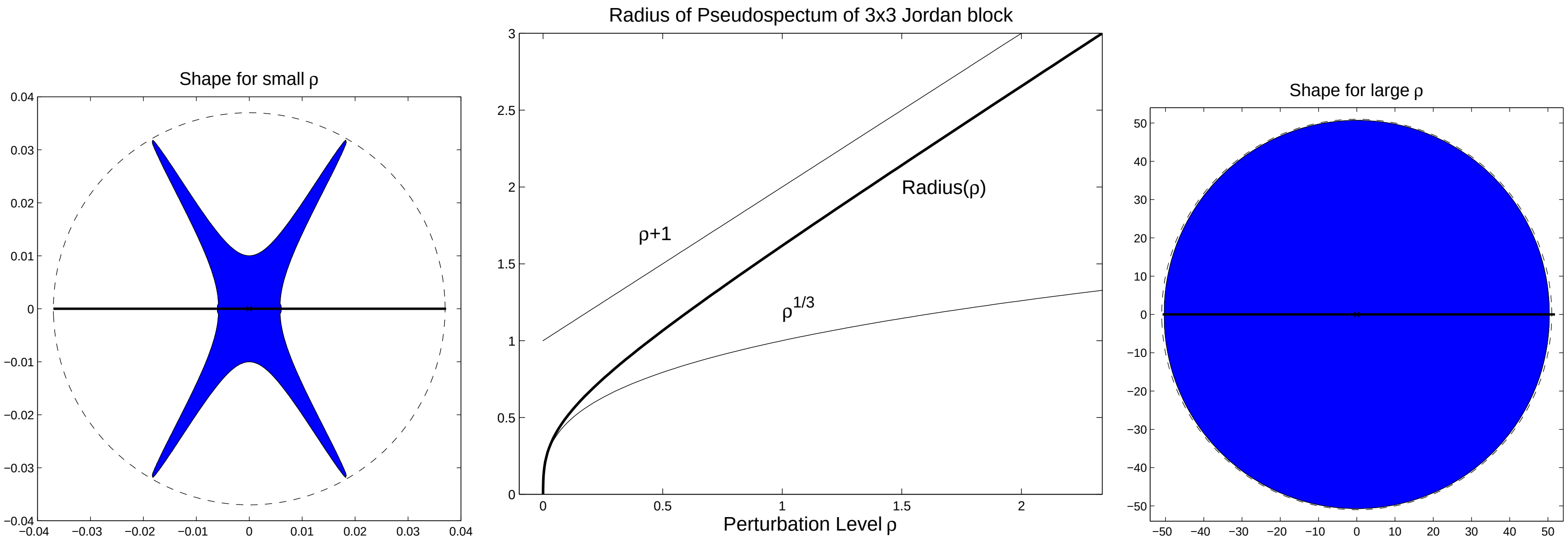
Perturbation level $\rho = 5 * 10^{-4}$

REAL pseudospectra of a REAL 3×3 Jordan block (w.r.t spectral norm)



Perturbation level $\rho = 5 * 10^{-5}$

Radius of the pseudospectra of a Jordan block



For an $n \times n$ Jordan block:

$$\lim_{\rho \to 0} \frac{Radius(\rho)}{\rho^{1/n}} = 1, \qquad \lim_{\rho \to \infty} (Radius(\rho) - \rho) = 1$$

Basic question:

How do the eigenvalues of a matrix A change by adding matrix perturbations Δ of a certain class struct and norm $\|\cdot\|$?

$$A \rightsquigarrow A + \Delta, \qquad \Delta \in \text{struct}, \; \|\Delta\| \leq \rho.$$

More generally (Hinrichsen, Pritchard) :

How do the eigenvalues of A change under perturbations of the form

$$A \rightsquigarrow A + B\,\Delta\,C, \qquad \Delta \in \text{struct}, \; \|\Delta\| \leq \rho,$$

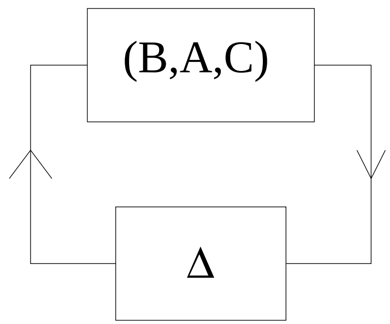
where $B \in \mathbb{C}^{n\times m}$ and $C \in \mathbb{C}^{p\times n}$ are fixed matrices ?

These are perturbations of feedback-form:

Open loop system: $\dot{x} = Ax + Bu, \quad y = Cx$

Feedback: $u = \Delta\, y$

Closed loop system: $\dot{x} = (A + B\Delta C)\,x$



Why B and C ?

These matrices can be used to model general linear perturbations of A

Examples:

- $$\begin{bmatrix} A_{11} & A_{12} + \triangle \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix}}_{B} \triangle \underbrace{\begin{bmatrix} 0 & I \end{bmatrix}}_{C}$$

- $$A + (\delta_1 E_1 + \delta_2 E_2) = A + \underbrace{\begin{bmatrix} I & I \end{bmatrix}}_{B} \begin{bmatrix} \delta_1 I & 0 \\ 0 & \delta_2 I \end{bmatrix} \underbrace{\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}}_{C}$$

Definition of μ-values: For $M \in \mathbb{C}^{m\times n}$, struct $\subseteq \mathbb{C}^{n\times m}$,

$$\mu_{\text{struct}}(M) := \left(\inf\{ \|\Delta\|;\ \Delta \in \text{struct},\ \det(\Delta M - I_n) = 0 \} \right)^{-1}.$$

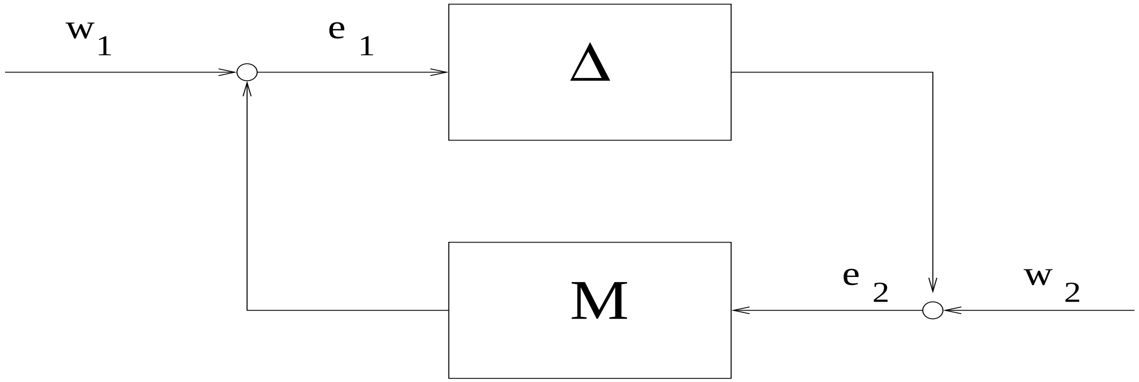
In words:

$1/\mu_{\text{struct}}(M)$ is the smallest norm of $\Delta \in$ struct s.t. 1 is an eigenvalue of the product ΔM.

Easy to see:

$$\text{struct} = \mathbb{C}^{n\times n},\ \|\cdot\| \text{ spectral norm} \quad \Rightarrow \quad \mu_{\text{struct}}(M) = \|M\| = \sigma_{\max}(M).$$

The quantity $\mu_{\text{struct}}(M)$ (**the structured singular value**) has been introduced by Doyle and Safonov in the 1980'ies as a tool for robustness analysis of linear systems.
($\rightsquigarrow$ μ-Toolbox of MATLAB)



Definition of μ again: For $M \in \mathbb{C}^{m\times n}$, struct $\subseteq \mathbb{C}^{n\times m}$,

$$\mu_{\text{struct}}(M) := \left(\inf\{ \|\Delta\|;\ \Delta \in \text{struct},\ \ 1 \in \sigma(\Delta\, M) \} \right)^{-1}.$$

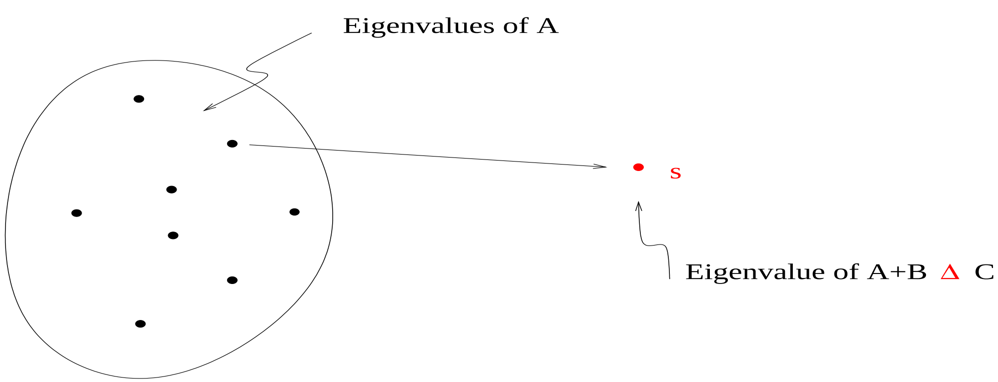
Main lemma:

For $s \in \mathbb{C}\backslash\sigma(A)$ let $G(s) = C(s - A)^{-1}B$. Then for all $\Delta \in \mathbb{C}^{n\times m}$,

$$1 \in \sigma(\Delta\, G(s)) \;\Leftrightarrow\; s \in \sigma(A + B\Delta C),$$

and hence

$$\frac{1}{\mu_{\text{struct}}(G(s))} = \inf\{ \|\Delta\|\,;\ \Delta \in \text{struct},\ s \in \sigma(A + B\Delta C) \}$$



Basic equivalence: For $s \notin \sigma(A)$,

$$1 \in \sigma(\Delta\, G(s)) \;\Leftrightarrow\; s \in \sigma(A + B\Delta C).$$

Proof:

$$\begin{aligned} s \in \sigma(A + B\Delta C) \;&\Leftrightarrow\; 0 = det(A - s + B\Delta C) \\ &\qquad = det((A - s)(I - (s - A)^{-1} B\Delta C)) \\ &\Leftrightarrow\; 0 = det(I - (s - A)^{-1} B\Delta C) \\ &\Leftrightarrow\; 1 \in \sigma((s - A)^{-1} B\Delta C) \\ &\Leftrightarrow\; 1 \in \sigma(\Delta\, \underbrace{C(s - A)^{-1} B}_{G(s)}). \end{aligned}$$

Application of μ: Stability radii

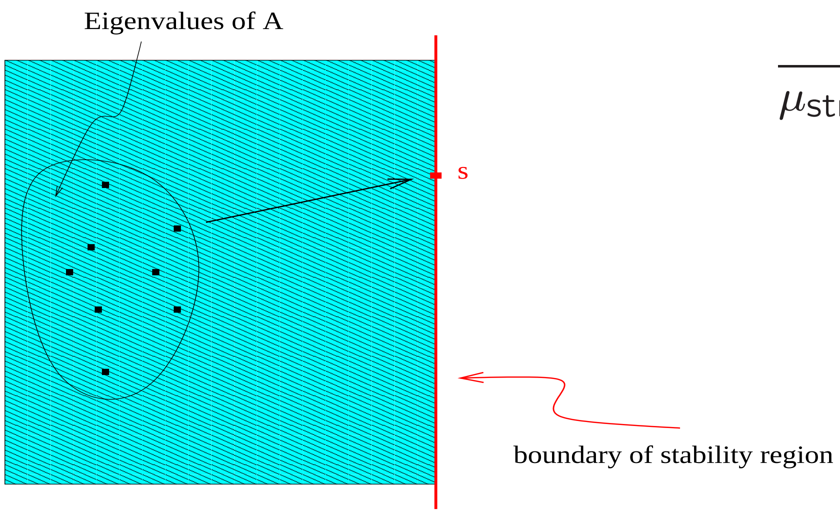
Definition: Stability radius with respect to open stability region $\mathbb{C}_g$:

$$r_{\text{struct}}(A,B,C,\mathbb{C}_g) := \inf\{\, \|\Delta\|; \quad \Delta \in \text{struct}, \;\; \sigma(A+B\Delta C) \not\subset \mathbb{C}_g \,\}.$$

Formula for computation: If $\sigma(A) \subset \mathbb{C}_g$ then

$$r_{\text{struct}}(A,B,C,\mathbb{C}_g) = \inf_{s\in\partial\mathbb{C}_g} \frac{1}{\mu_{\text{struct}}(G(s))} = \frac{1}{\sup_{s\in\partial\mathbb{C}_g} \mu_{\text{struct}}(G(s))}.$$

Proof: This is an immediate consequence of the main lemma



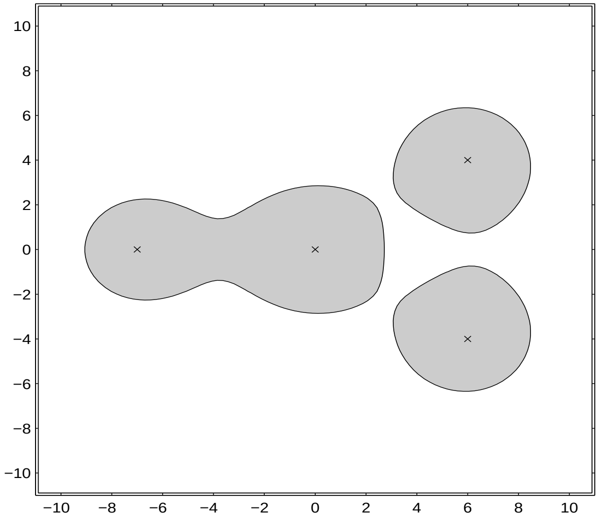
$$\frac{1}{\mu_{\text{struct}}(G(s))} = \inf\{\, \|\Delta\| \,; \quad \Delta \in \text{struct}, \quad s \in \sigma(A+B\Delta C) \,\}$$

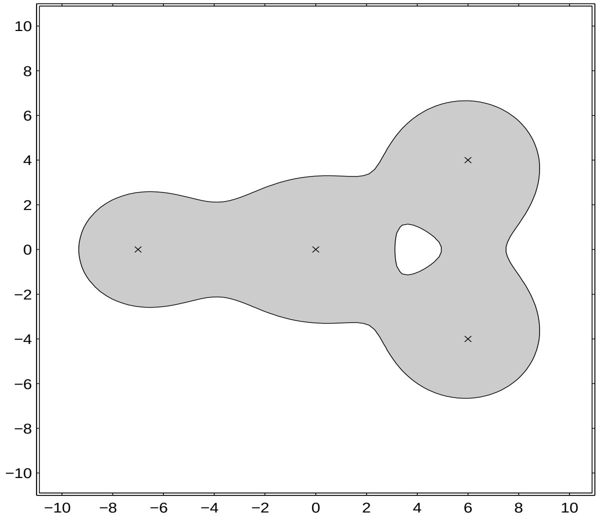
Spectral value sets (structured Pseudospectra)

Definition: Let $(A, B, C) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times p} \times \mathbb{C}^{q \times n}$.

$\sigma_{\text{struct}}(A, B, C, \rho)$:= Set of eigenvalues of all matrices of the form

$$A + B \Delta C, \qquad \Delta \in \text{struct}, \ \|\Delta\| \leq \rho.$$





We have (see e.g. Hinrichsen and Pritchard, 2005)

$$\sigma_{\text{struct}}(A, B, C, \rho) = \sigma(A) \cup \{ \, s \in \mathbb{C}; \ \ \mu_{\text{struct}}(\, C(sI_n - A)^{-1}B \,) \geq \rho^{-1} \, \}.$$

Unstructured Complex μ for Operator Norms

Definition of $\mu_{\mathbb{C}}$:

$$\mu_{\mathbb{C}}(M) := 1/\min\{\ \|\Delta\| \ \mid\ \Delta \in \mathbb{C}^{\ell\times q},\ 1 \in \sigma(\Delta M)\ \}$$

Lemma: (well known)

$$\|\Delta\| = \|\Delta\|_{\alpha,\beta} = \max_{x\neq 0}\frac{\|\Delta x\|_\beta}{\|x\|_\alpha} \quad \Rightarrow \quad \boxed{\mu_{\mathbb{C}}(M) = \|M\|_{\beta,\alpha} = \max_{x\neq 0}\frac{\|Mx\|_\alpha}{\|x\|_\beta}}$$

Definition of μ: For $M \in \mathbb{C}^{m\times n}$, struct $\subseteq \mathbb{C}^{n\times m}$,

$$\mu_{\text{struct}}(M) := \left(\inf\{ \|\Delta\|; \ \Delta \in \text{struct}, \ 1 \in \sigma(\Delta M) \} \right)^{-1}.$$

μ with respect to spectral norm:

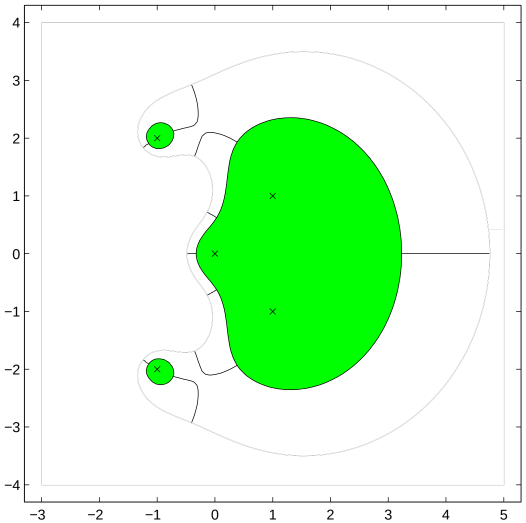
Complex Case:	**Real Case:**
(well known)	**(Bernhardsson, Davisson, Doyle, Qiu, Rantzer, Young, ca. 1993)**
$\mu_{\mathbb{C}}(M) = \sigma_1(M)$	$\mu_{\mathbb{R}}(M) = \inf_{\gamma\in(0,1]} \sigma_2\left(\begin{bmatrix} \Re M & -\gamma \Im M \\ \gamma^{-1} \Im M & \Re M \end{bmatrix}\right)$

For the computation of μ for Hermitian, (skew)symmetric and Hamiltonian perturbations see

Michael Karow.
μ-values and spectral value sets for linear perturbation classes defined by a scalar product.
MATHEON-Preprint 406. (2007)

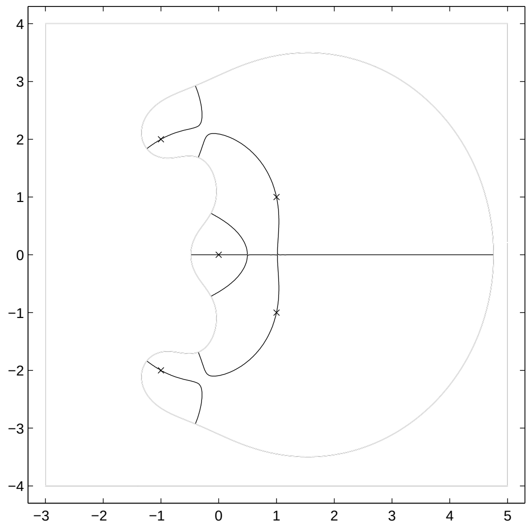
Some real Spectral value sets

$$\sigma_{\mathbb{R}}(A,B,C,\rho) = \sigma(A) \cup \{\, s \in \mathbb{C} \mid \mu_{\mathbb{R}}(G(s)) \geq \rho^{-1} \,\}, \qquad G(s) = C(sI - A)^{-1}B.$$

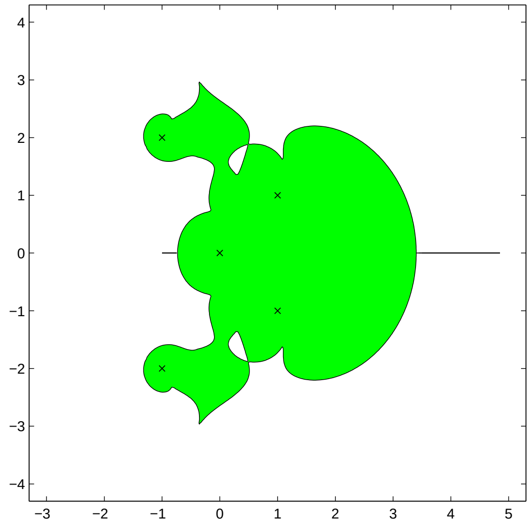


$$G_1(s) := g(s)\,\mathsf{diag}(1, \tfrac{1}{4}),$$

$$g(s) := \frac{((s+\frac{1}{2})^2+1)((s+2)^2+1)}{s((s-1)^2+1)((s+1)^2+4)},$$



$$G_2(s) := g(s)\,\mathsf{diag}(1, 0),$$



$$G_3(s) := \mathsf{diag}(\, g(s), h(s) \,),$$

$$h(s) := \frac{((s+\frac{1}{2})^2+1)((s+2)^2+1)+40}{s((s-1)^2+1)((s+1)^2+4)}$$

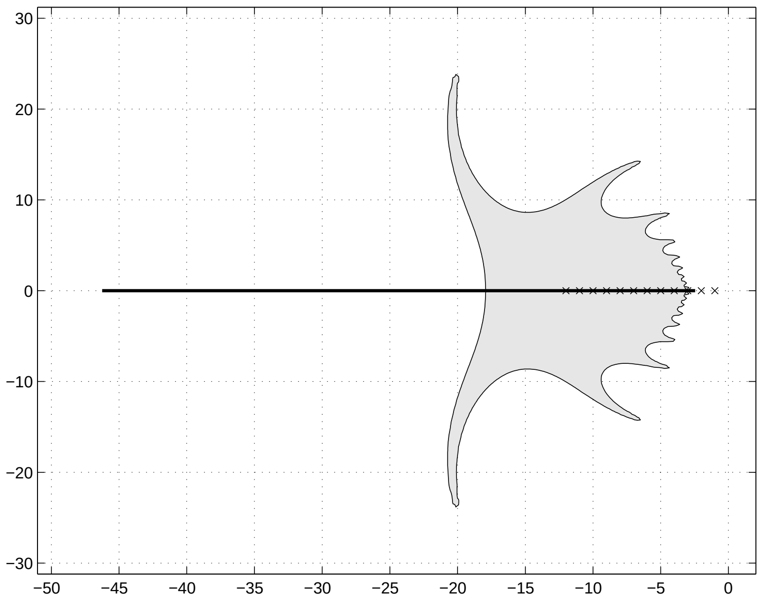
Example: Root sets of an interval polynomial

Center polynomial: $$p(s) = s^{12} - \sum_{k=0}^{11} a_k s^k = \prod_{j=1}^{12} (s + j)$$

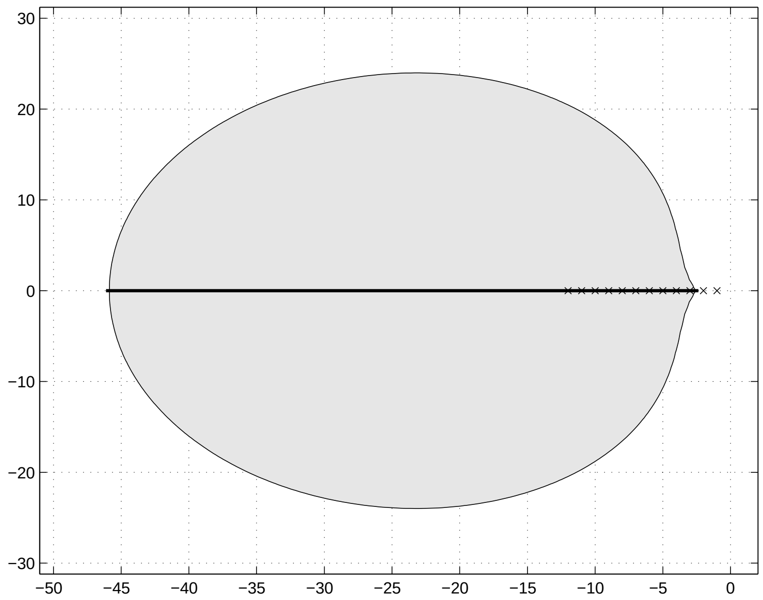
The figures show the sets

$$\{\ s \in \mathbb{C} \mid \ s^n - \sum_{k=0}^{n-1} (a_k + \Delta_k) s^k = 0, \quad \Delta_k \in \mathbb{F}, \ |\Delta_k| \le 7 \ \}$$

real perturbations



complex perturbations

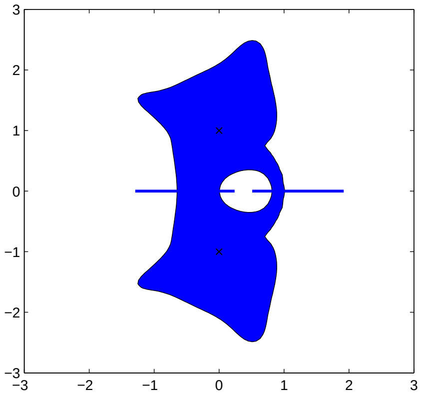


Example: The sets

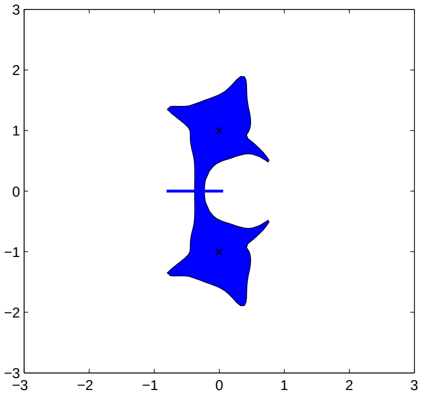
$$\sigma_{\mathbb{R}}(A,B,C,\rho) = \bigcup_{\Delta \text{ real}, \ \|\Delta\|_2 \le \rho} \sigma\left(\begin{bmatrix} \Re(\lambda I_n + M) + \Delta & -\Im(\lambda I_n + M) \\ \Im(\lambda I_n + M) & \Re(\lambda I_n + M) \end{bmatrix} \right)$$

where $\lambda = i$ and $M = \begin{bmatrix} 0 & Z \\ 0 & 0 \end{bmatrix}$, $\quad Z = \begin{bmatrix} -2+2i & -3+3i \\ -2+3i & -4+5i \end{bmatrix}$

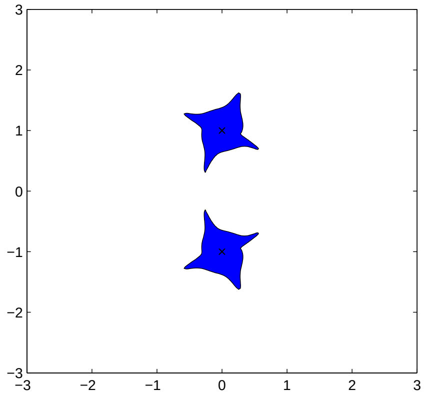
$\rho = 0.5$



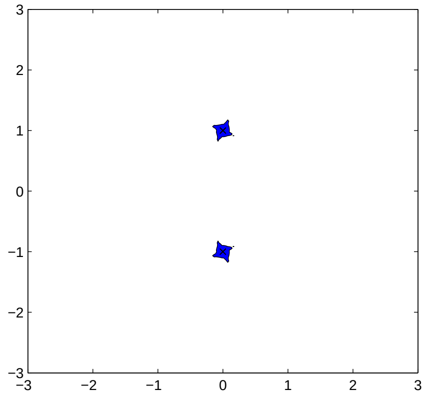
$\rho = 0.2$



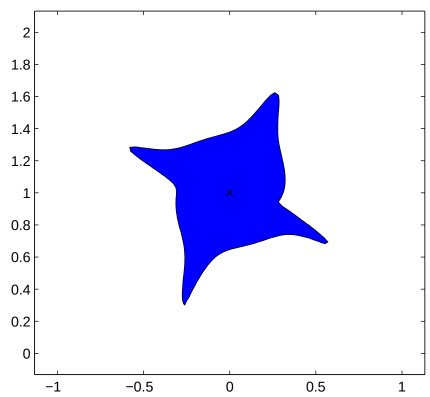
$\rho = 0.1$

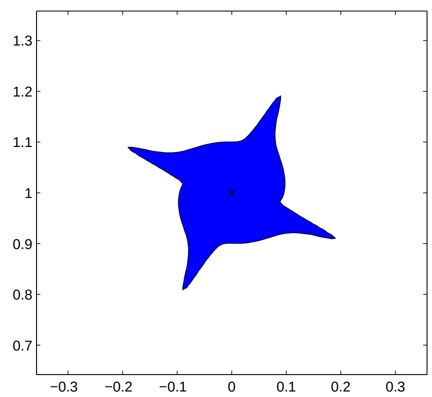


$\rho = 0.01$



$\downarrow$ zoom $\rho^{-1/2}$ $\qquad$ $\downarrow$ zoom $\rho^{-1/2}$





Partial Fraction Expansion of Transfer Function

Jordan decomposition: $$A = \sum_{\lambda\in\sigma(A)} (\lambda P_\lambda + N_\lambda)$$

Partial fraction expansion of resolvent (i_λ = index of nilpotency):

$$(sI_n - A)^{-1} = \sum_{\lambda\in\sigma(A)} \left(\frac{P_\lambda}{s-\lambda} + \sum_{k=2}^{i_\lambda} \frac{N_\lambda^{k-1}}{(s-\lambda)^k} \right).$$

Hence, we have for the transfer function $G(s) = C(sI - A)^{-1}B$:

$$\boxed{G(s) = \sum_{\lambda\in\sigma(A)} \left(\frac{CP_\lambda B}{s-\lambda} + \sum_{k=2}^{k_\lambda} \frac{CN_\lambda^{k-1}B}{(s-\lambda)^k} \right)}$$

Notation for leading coefficients:

$$0 \neq \Gamma_\lambda := \begin{cases} CP_\lambda B & \text{if } k_\lambda = 1, \\ CN_\lambda^{k_\lambda - 1} B & \text{otherwise.} \end{cases}$$

Main result (K., 2008)

Let $\mathcal{C}_\lambda(\rho)$ denote the connected component of $\sigma_{\text{struct}}(A, B, C, \rho)$ that contains the eigenvalue $\lambda \in \sigma(A)$.
Then we have with respect to the **Hausdorff metric**,

$$\lim_{\rho \searrow 0} \frac{\mathcal{C}_\lambda(\rho) - \lambda}{\rho^{1/k_\lambda}} = \mathcal{L}_\lambda,$$

where

$$\begin{aligned}
\mathcal{L}_\lambda &:= \{\, s \in \mathbb{C};\ \ s^{k_\lambda} \in \sigma(\Delta\Gamma_\lambda) \text{ for some } \Delta \in \text{struct with } \|\Delta\| \leq 1 \,\} \\
&= \text{set of roots of order } k_\lambda \text{ of} \\
&\quad\ \text{the eigenvalues of } \Delta\Gamma_\lambda,\ \Delta \in \text{struct with } \|\Delta\| \leq 1 \\
&= \{\, r\, e^{i\phi} \mid \phi \in [0, 2\pi),\ 0 \leq r \leq R_\lambda(\phi) \,\},
\end{aligned}$$

where

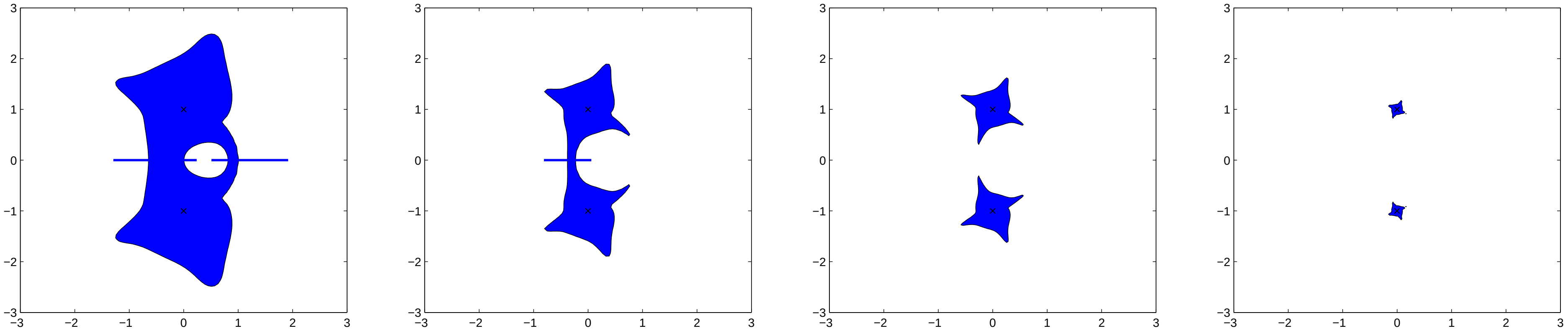
$$R_\lambda(\phi) := [\, \mu_{\text{struct}}(e^{-ik_\lambda\phi}\Gamma_\lambda) \,]^{1/k_\lambda}.$$

Example continued: The sets

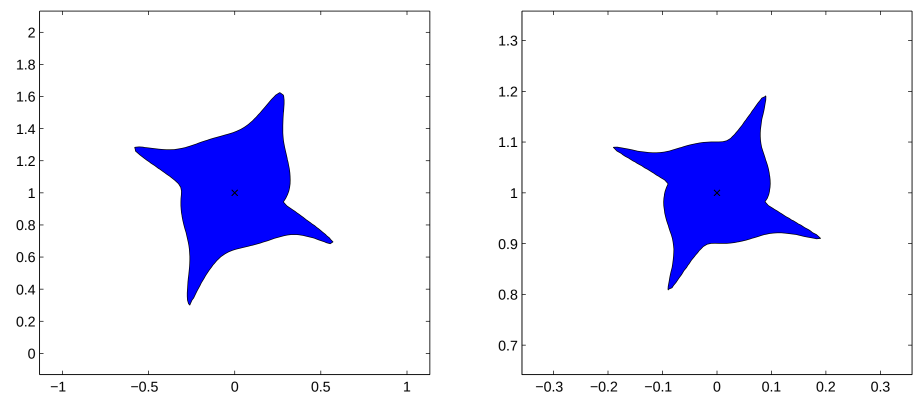
$$\sigma_{\mathbb{R}}(A,B,C,\rho) = \bigcup_{\Delta \text{ real},\ \|\Delta\|_2 \le \rho} \sigma\left(\begin{bmatrix} \Re(\lambda I_n + M) + \Delta & -\Im(\lambda I_n + M) \\ \Im(\lambda I_n + M) & \Re(\lambda I_n + M) \end{bmatrix}\right)$$

where $M = \begin{bmatrix} 0 & Z \\ 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} -2+2i & -3+3i \\ -2+3i & -4+5i \end{bmatrix}$

$\rho = 0.5$ $\quad$ $\rho = 0.2$ $\quad$ $\rho = 0.1$ $\quad$ $\rho = 0.01$



$\downarrow$ zoom $\rho^{-1/2}$ $\quad$ $\downarrow$ zoom $\rho^{-1/2}$

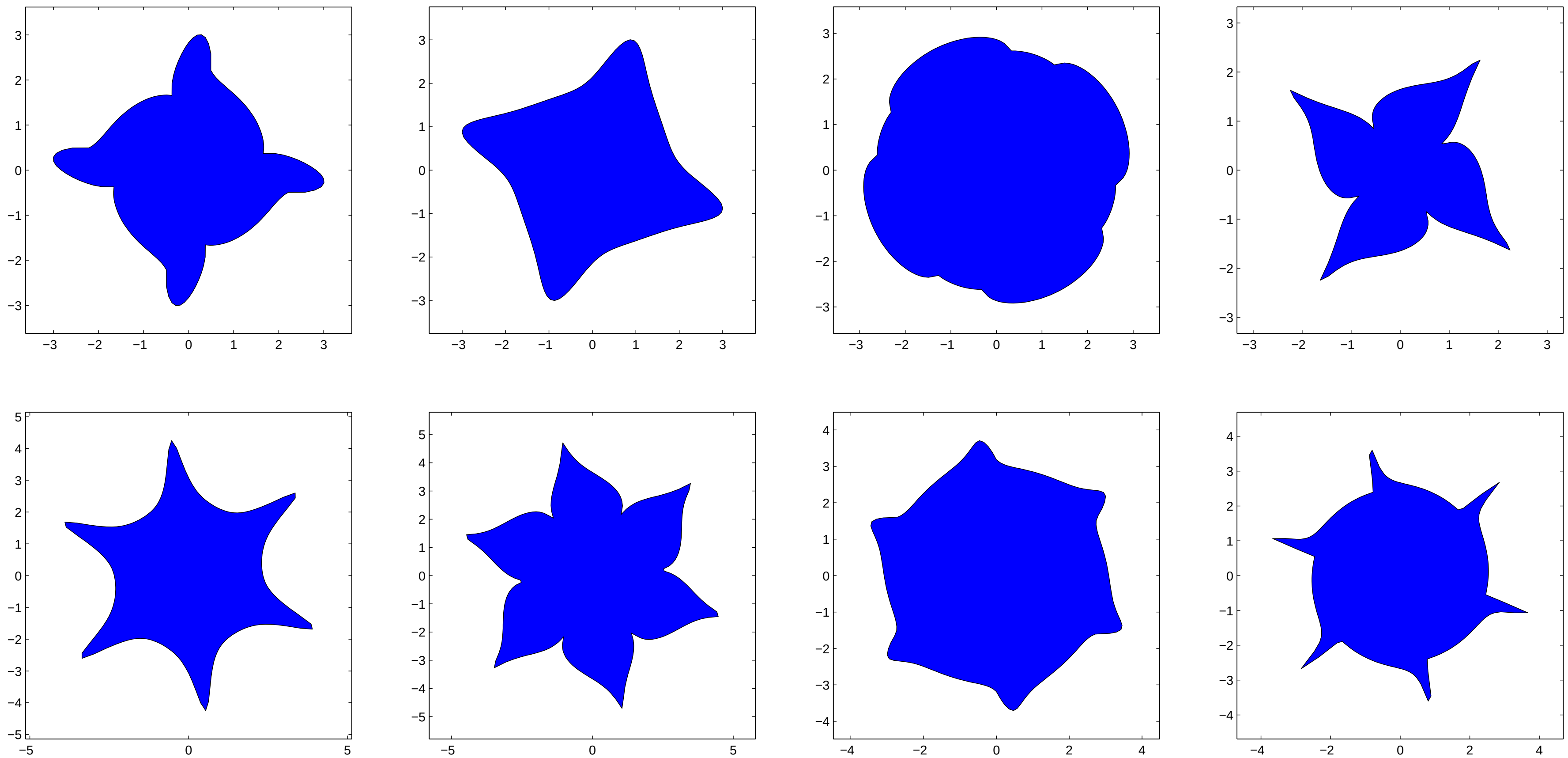


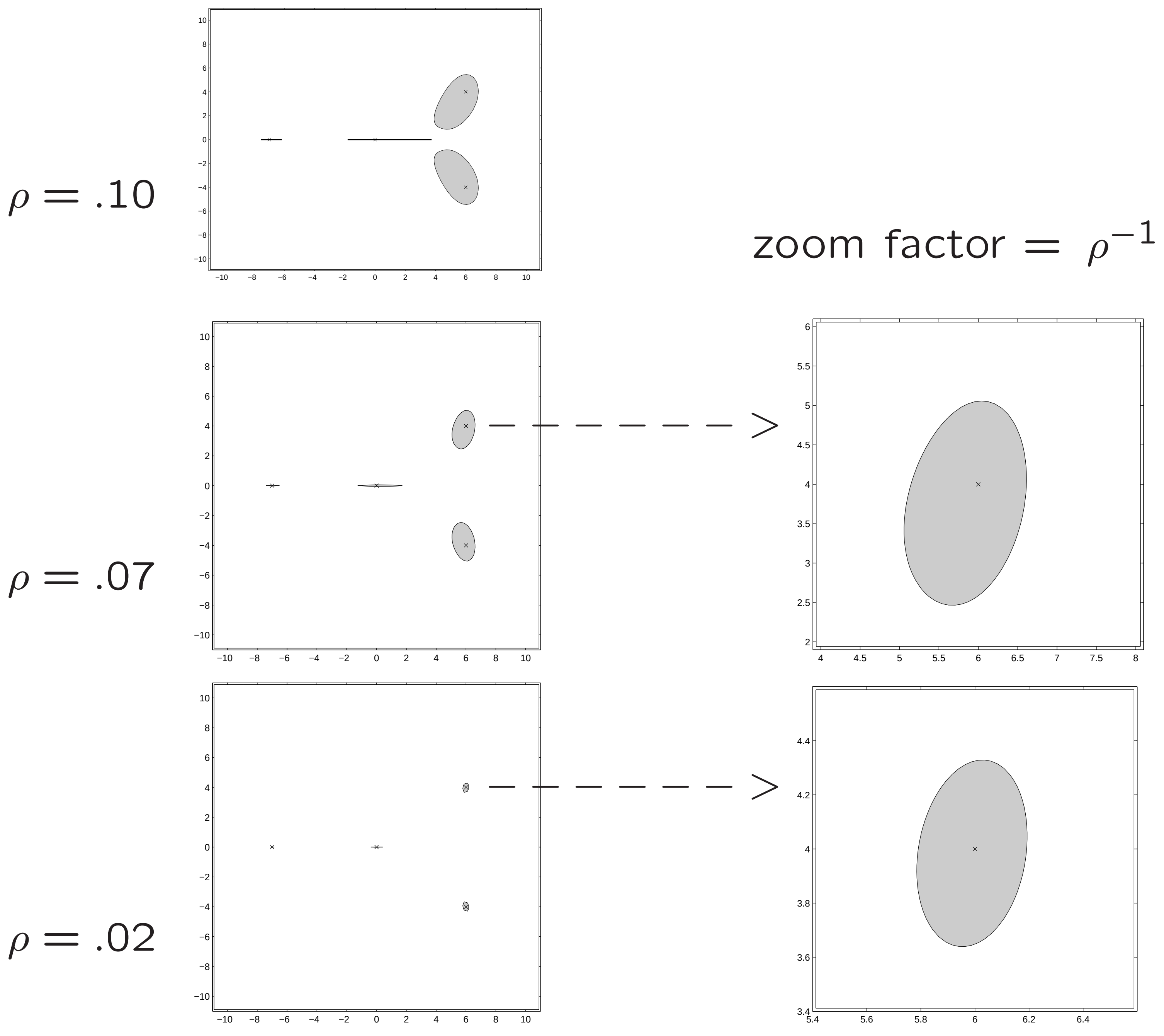
Example continued: The figures below show some limit sets

$$\mathcal{L}_\lambda = \{\, s \in \mathbb{C};\ \ s^i \in \sigma(\Delta\Gamma_\lambda) \text{ for some } \Delta \in \mathbb{R}^{n\times n} \text{ with } \|\Delta\|_2 \le 1 \,\}, \qquad \Gamma_\lambda = M^{i-1}$$

for $M = \begin{bmatrix} 0 & Z \\ 0 & 0 \end{bmatrix}$ (upper row) and $M = \begin{bmatrix} 0 & Z & 0 \\ 0 & 0 & Z \\ 0 & 0 & 0 \end{bmatrix}$ (lower row)

where $Z \in \mathbb{C}^{2\times 2}$ is chosen at random.





$\rho = .10$
zoom factor = ρ^{-1}
$\rho = .07$
$\rho = .02$

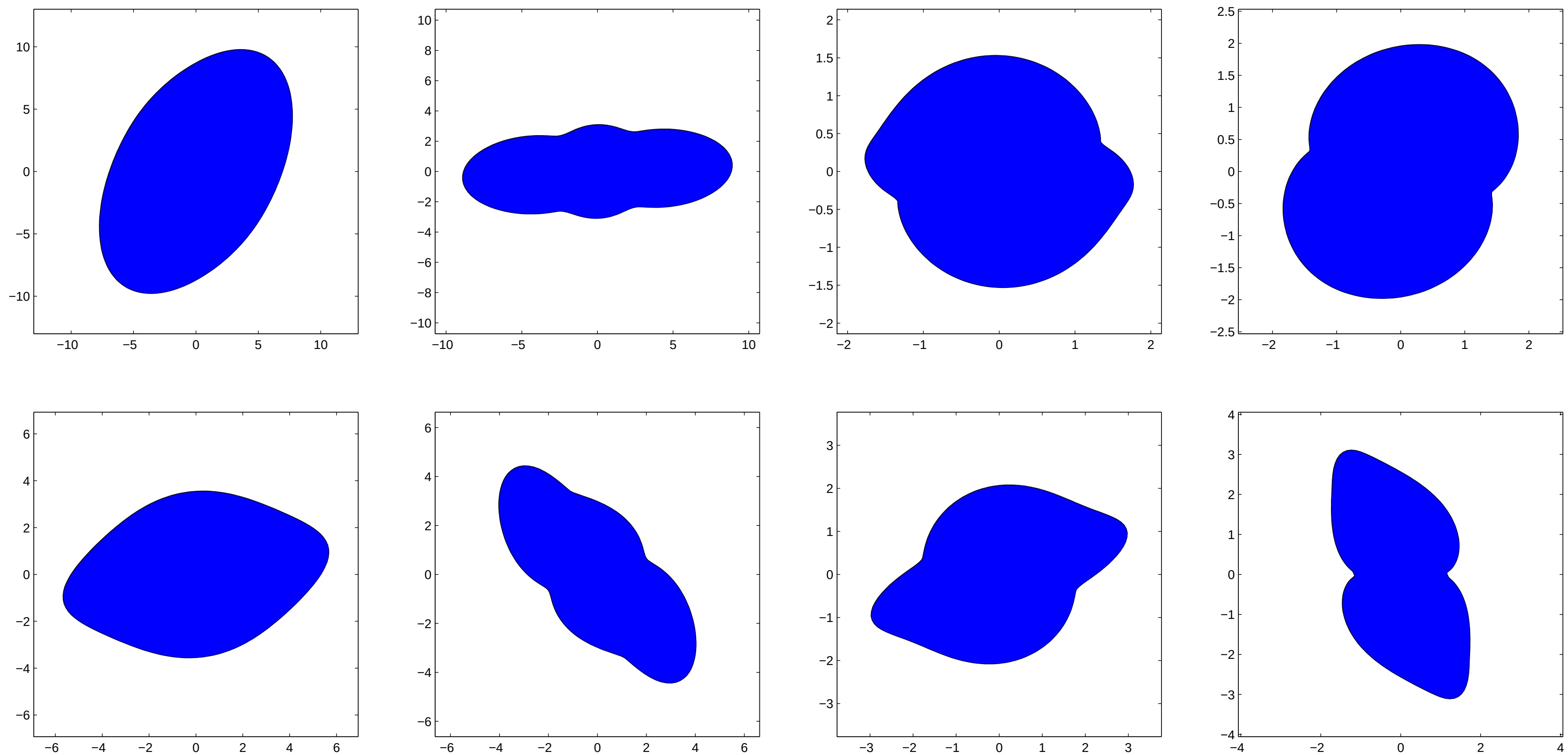
Some limit sets for semisimple nonreal eigenvalues

Jordan decompositon: $A = \lambda P_\lambda + \bar{\lambda} \bar{P_\lambda} + \ldots, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}.$

Limit sets:

$$\mathcal{L}_\lambda = \{ s \in \mathbb{C};\ \ s \in \sigma(\Delta P_\lambda) \text{ for some } \Delta \in \mathbb{R}^{n \times n},\ \ \|\Delta\|_2 \leq 1 \}.$$

The figures show some limit sets for the case $P_\lambda \in \mathbb{C}^{4\times 4}$, $\text{rank}(P_\lambda) = 2$.



Interconnected linear systems

M. Karow, D. Hinrichsen, A. J. Pritchard.

Interconnected systems with uncertain couplings:
explicit formulae for μ-values, spectral value sets and stability radii.

SIAM J. Control Opt. 45(3):856-884, 2006.

Interconnected systems:

Linear Systems: $\dot{x}_j = A_j x_j + B_j u_j, \qquad y_j = C_j x_j \qquad j = 1, \ldots, p$

Couplings: $u_j = \Delta_{j1} y_1 + \ldots + \Delta_{jp} y_p$

Interconnected system:

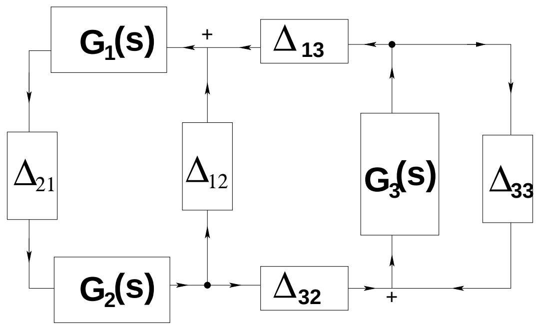
$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_p \end{bmatrix} = A_\Delta \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}, \qquad A_\Delta = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{bmatrix} + \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_p \end{bmatrix} \underbrace{\begin{bmatrix} \Delta_{11} & \ldots & \Delta_{1p} \\ \vdots & & \vdots \\ \Delta_{p1} & \ldots & \Delta_{pp} \end{bmatrix}}_{=:\Delta} \begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_p \end{bmatrix}$$

Problems: Where in the complex plane can the eigenvalues of A_Δ lie if

(1) some prescribed blocks Δ_{jk} are zero

(2) $\|\Delta\| = \mathcal{N}(\|\Delta_{11}\|, \ldots, \|\Delta_{pp}\|)$ is bounded by $\delta > 0$

Example:



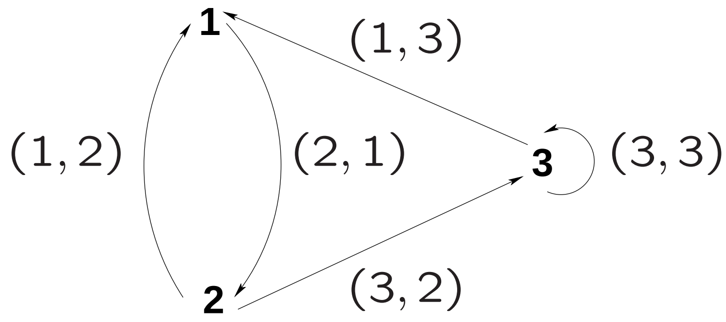
$G_j(s) = C_j(s - A_j)^{-1} B_j$

Example: An interconnected system

index set

$$\mathcal{I} = \{\, (1,2),\ (1,3) \\ (2,1), \\ (3,2), (3,3) \}$$

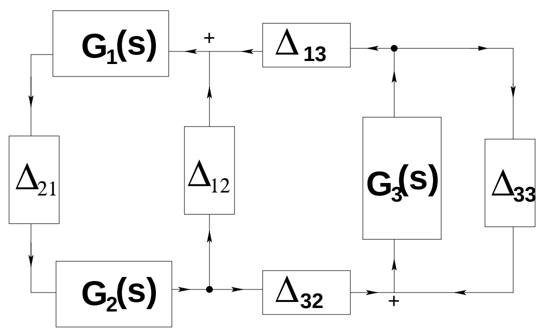
directed graph $\Gamma(\underline{3}, \mathcal{I})$



interconnection structure

$$\begin{bmatrix} 0 & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & 0 & 0 \\ 0 & \Delta_{32} & \Delta_{33} \end{bmatrix} \in \text{struct}_{\mathcal{I}}$$

block diagram



Pert. class: $\quad \text{struct} = \text{struct}_{\mathcal{I}} = \{\ [\Delta_{jk}];\ \ \Delta_{jk} = 0 \text{ if } (j,k) \notin \mathcal{I}\ \}$

Interconnected system again:

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_p \end{bmatrix} = A_{\Delta} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}, \quad A_{\Delta} = \underbrace{\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_p \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \Delta_{11} & \dots & \Delta_{1p} \\ \vdots & & \vdots \\ \Delta_{p1} & \dots & \Delta_{pp} \end{bmatrix}}_{=:\Delta} \underbrace{\begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_p \end{bmatrix}}_{C}$$

Needed for eigenvalue perturbation analysis:

$$\mu_{\mathsf{struct}}(\, C(s-A)^{-1}B \,) \;=\; \mu_{\mathsf{struct}}(\, \mathsf{diag}(G_1(s), G_2(s), \dots, G_p(s)) \,),$$

where

$$G_j(s) = C_j(s - A_j)^{-1}B_j.$$

μ-value depends on

- the interconnection structure, i.e. the perturbation class

$$\mathsf{struct} = \mathsf{struct}_{\mathcal{I}} = \{\ [\Delta_{jk}];\ \ \Delta_{jk} = 0 \text{ if } (j,k) \notin \mathcal{I}\ \},$$

- the underlying norm $\|\Delta\|$.

Theorem (K., Hinrichsen, Pritchard 2006):
Let $\mathcal{I} \subseteq \{1,\ldots,p\}^2$ be an index set and struct be the set of all block matrices of the form

$$\Delta = \begin{bmatrix} \Delta_{11} & \cdots & \Delta_{1p} \\ \vdots & & \vdots \\ \Delta_{p1} & \cdots & \Delta_{pp} \end{bmatrix} \qquad \text{with } \Delta_{jk} = 0 \text{ for } (j,k) \notin \mathcal{I}$$

(fixed block sizes)

Let $\mathcal{N} = \|\cdot\|_{\alpha,\alpha}$ be an operator norm on $\mathbb{R}^{p\times p}$ and

$$\|\Delta\| := \mathcal{N}\left(\begin{bmatrix} \|\Delta_{11}\|_{\beta_1,\alpha_1} & \cdots & \|\Delta_{1p}\|_{\beta_1,\alpha_p} \\ \vdots & & \vdots \\ \|\Delta_{p1}\|_{\beta_p,\alpha_1} & \cdots & \|\Delta_{pp}\|_{\beta_p,\alpha_p} \end{bmatrix} \right)$$

for all $\Delta \in$ struct with operator norms $\|\cdot\|_{\beta_k,\alpha_k}$.

Then for all $M = \text{diag}(M_1,\ldots,M_p)$:

$$\mu^{\|\cdot\|}_{\text{struct}}(M) = \max_{\gamma\in Z(\mathcal{I})} \left(\prod_{j\in\gamma} \|M_j\|_{\alpha_j,\beta_j} \right)^{\frac{1}{|\gamma|}},$$

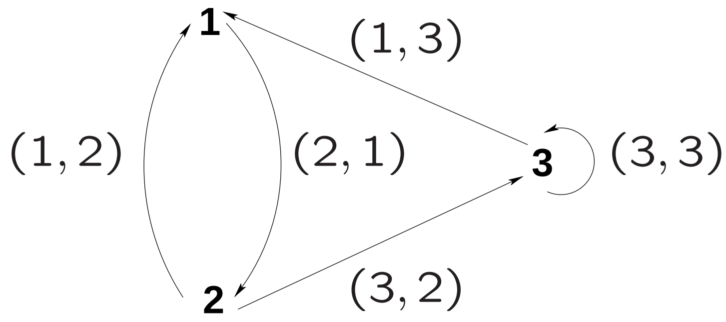
where $Z(\mathcal{I})$ denotes the set of **cycles** in the directed Graph associated with $\mathcal{I}$.
$|\gamma|$ is the length of the cycle γ.

Example: An interconnected system

index set

$$\mathcal{I} = \{\, (1,2),\ (1,3) \\ (2,1), \\ (3,2), (3,3)\}$$

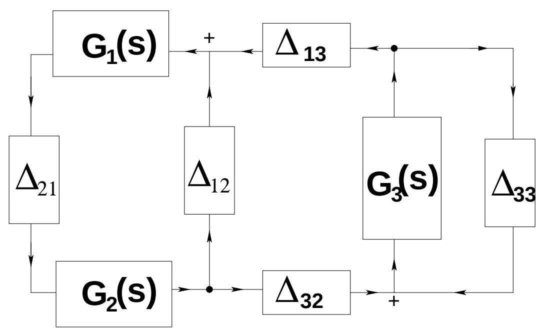
directed graph $\Gamma(\underline{3}, \mathcal{I})$



interconnection structure

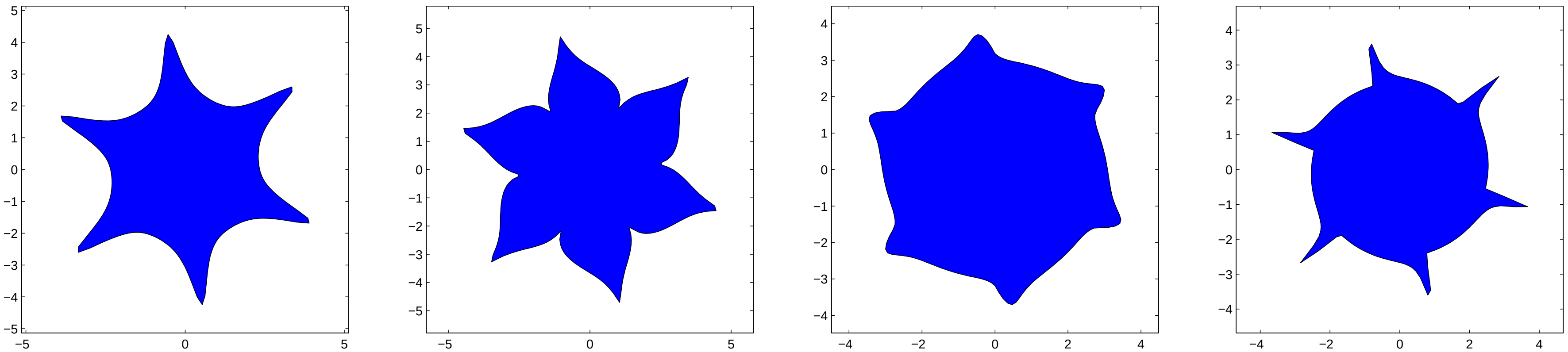
$$\begin{bmatrix} 0 & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & 0 & 0 \\ 0 & \Delta_{32} & \Delta_{33} \end{bmatrix} \in \mathrm{struct}_{\mathcal{I}}$$

block diagram



Pert. class: $\quad \mathrm{struct} = \mathrm{struct}_{\mathcal{I}} = \{\ [\Delta_{jk}];\ \ \Delta_{jk} = 0 \text{ if } (j,k) \notin \mathcal{I}\ \}$

Thanks for listening



Question: How do the eigenvalues of A change under perturbations of feedback form?

$$A \rightsquigarrow A + B \Delta C, \qquad \Delta \in \text{struct}, \; \|\Delta\| \leq \rho. \qquad (*)$$

Related mathematical objects:

- Structured eigenvalue condition numbers
  (they measure the change of eigenvalues for small perturbations)

- Spectral value sets (structured pseudospectra)
  (Sets of all eigenvalues of all matrices of the form $(*)$)

- Stability radii
  (smallest norm of a perturbation Δ such that $(*)$ becomes unstable)

All these quantities can be computed via

- μ-values
- the structured distances of matrices to the set of singular matrices ($\tilde{\mu}$-values)