GIAN course: Singular optimal control

Slide collection 8

The Stability radius and an application of KYP

Michael Karow, TU Berlin, December 2016

karow@math.tu-berlin.de

The stability radius

Linear system: $\dot{x} = A\,x + Bu, \quad y = Cx$

Output feedback: $u = \Delta\, y$ $\qquad$ (Δ = feedback matrix)

Closed loop: $\dot{x} = \underbrace{(A + B\,\Delta C)}_{=:A_\Delta}\, x$

Problem (distance to instability): Suppose A is stable (all eigenvalues in left half plane).

What is the smallest Δ, such that A_Δ is unstable? Small means $\|\Delta\|_2$ small.

Stability radius:

$$\begin{aligned} r_{\mathbb{C}}(A,B,C) &:= \inf\{\|\Delta\|_2 \mid \Delta \in \mathbb{C}^{p\times q},\ A_\Delta \text{ unstable}\,\} \\ &= \sup\{\rho > 0 \mid A_\Delta \text{ stable for all } \Delta \in \mathbb{C}^{p\times q} \text{ with } \|\Delta\|_2 < \rho\,\}. \end{aligned}$$

Since eigenvalues of A_Δ depend continuously on Δ:

$$r_{\mathbb{C}}(A,B,C) = \inf\{\rho > 0 \mid \exists\Delta : (\|\Delta\|_2 = \rho \text{ and } A_\Delta \text{ has eigenvalue on imaginary axis})\}$$

Moving eigenvalues by minimal perturbations

Lemma. 1 Suppose $\lambda \in \mathbb{C}$ is not an eigenvalue of A. Then

$$\lambda \text{ is eigenvalue of } A_\Delta = A + B\Delta C \quad \Leftrightarrow \quad \exists v \neq 0 : \Delta\, G(\lambda)\, v = v,$$

where

$$G(\lambda) = C(\lambda I - A)^{-1}B \qquad \text{(transfer function)}$$

Proof. If λ is eigenvalue then there is $w \neq 0$ such that

$$\begin{aligned}(A + B\Delta C)w = \lambda w \quad &\Rightarrow \quad B\Delta C w = (\lambda I - A)w \\ &\Rightarrow \quad (\lambda I - A)^{-1} B\Delta C w = w \qquad (w \neq 0 \text{ implies } \Delta C w \neq 0) \\ &\Rightarrow \quad \underbrace{\Delta C(\lambda I - A)^{-1}B}_{G(\lambda)} \underbrace{\Delta C w}_{v} = \underbrace{\Delta C\, w}_{v}.\end{aligned}$$

Proof of other direction analogous: if $\Delta\, G(\lambda)\, v = v$ then $w = (\lambda I - A)^{-1}Bv$ is eigenvector.

Lemma 2. If λ is eigenvalue of A_Δ then since $\Delta\, G(\lambda)v = v$ we have $\|\Delta\|_2 \|G(\lambda)\|_2 \geq 1$.

Lemma 3. A smallest Δ such that A_Δ has eigenvalue λ is given by

$$\Delta := \frac{v\, G(\lambda)^*}{\|G(\lambda)\|_2^2},$$

where v is a normalized singular vector to the largest singular value of $G(\lambda)$:

$$\|v\|_2 = 1, \quad \|G(\lambda)v\|_2 = \|G(\lambda)\|_2^2, \quad \|\Delta\|_2 = \frac{1}{\|G(\lambda)\|_2}.$$

Formula for the complex stability radius $r_{\mathbb{C}}(A,B,C)$

From the lemmas on the slide before we have:

If $\lambda \in \mathbb{C}$ is not an eigenvalue of A then there exists a matrix Δ with

$$\|\Delta\|_2 = \frac{1}{\|G(\lambda)\|_2} = \frac{1}{\|C(\lambda I - A)^{-1}B\|_2}$$

such that λ is an eigenvalue of $A_\Delta = A + B\Delta C$.
There is no matrix of smaller norm with the latter property.

Consequence for stability radius of stable A:

$$\begin{aligned} r_{\mathbb{C}}(A,B,C) &= \inf\{\rho > 0 \mid \exists \Delta : (\|\Delta\|_2 = \rho \text{ and } A_\Delta \text{ has eigenvalue on imaginary axis})\} \\ &= \inf\{\ \|G(i\omega)\|^{-1} \mid \omega \in \mathbb{R}\ \} \\ &= \frac{1}{\sup_{\omega\in\mathbb{R}} \|G(i\omega)\|_2} \end{aligned}$$

Stability of nonlinear feedback
(here the KYP-Lemma comes into play)

We considered stability for linear systems of the form

$$\dot{x}(t) = (A + B\Delta C)\, x(t).$$

Replacing here the linear map $y \mapsto \Delta\, y$ by an arbitrary function f we obtain the ODE

$$\dot{x}(t) = A\, x(t) + B\, f(t, C\, x(t)).$$

Question: under which condition is

$$\lim_{t \to \infty} x(t) = 0 \;\; ?$$

Conjecture: sufficient condition is

$$\|f(t, y)\|_2 \leq r\, \|y\|_2, \qquad \text{for some} \qquad r < r_{\mathbb{C}}(A, B, C).$$

Conjecture is true. The proof on following pages via KYP-Lemma.

Theorem. Suppose the function $(t,y) \mapsto f(t,y)$ is locally Lipschitz with respect to y (sufficient cond. for local existence of solutions of ODE) and satisfies

$$\|f(t,y)\|_2 \le r\,\|y\|_2 \qquad \text{for some } r < r_{\mathbb{C}}(A,B,C).$$

Suppose further, that A is (Hurwitz) stable. Then

$$\dot{x}(t) = A\,x(t) + B\,f(t, C\,x(t)) \qquad \Rightarrow \qquad \lim_{t\to\infty} x(t).$$

Proof.

$$r < r_{\mathbb{C}}(A,B,C) = \frac{1}{\sup_{\omega\in\mathbb{R}} \|G(i\,\omega)\|_2} \qquad \Rightarrow \qquad r^{-2} > \sup_{\omega\in\mathbb{R}} \|G(i\,\omega)\|_2^2$$

Thus, for all $\omega \in \mathbb{R}$,

$$0 \; < \; r^{-2}\,I - G(i\omega)^* G(i\,\omega) \; = \begin{bmatrix} (i\omega\,I - A)B \\ I \end{bmatrix}^* \begin{bmatrix} -C^*C & 0 \\ 0 & r^{-2} \end{bmatrix} \begin{bmatrix} (i\omega\,I - A)B \\ I \end{bmatrix} \qquad \text{(Popov func.)}$$

By KYP there exists $P = P^*$ such that

$$0 \; < \; \begin{bmatrix} A^*P + PA - C^*C & PB \\ B^*P & -r^{-2}\,I \end{bmatrix}$$

Taking the Schur complement we conclude

$$(A^*P + PA - C^*C) - r^2\,PBB^*P > 0 \qquad \text{(Riccati inequality)}$$

This implies $A^*P + PA > 0$, and hence since A is stable: $\quad P < 0$.

Proof continued: The condition $\|f(y,t)\|_2 \le r\,\|y\|_2$ implies

$$\|f(t,Cx(t))\|_2^2 \le r^2\,\|Cx\|_2^2 = r^2\,x^*C^*Cx.$$

In the following computation $f(t,Cx(t))$ if abbreviated by f. With this notation we have

$$r^{-2}\,\|f\|^2 \le x^*C^*Cx \qquad (*)$$

From $\dot{x} = Ax + Bf$ it follows that

$$\begin{aligned}
\frac{d}{dt}x^*Px &= x^*P\dot{x} + \dot{x}^*Px \\
&= x^*P(Ax + B\,f) + (Ax + B\,f)^*Px \\
&= f^*(B^*Px) + ((B^*Px)^*f + x^*(A^*P + PA)x \\
&= \|r\,B^*Px + r^{-1}\,f\|_2^2 - r^2\,x^*PBB^*Px - \underbrace{r^{-2}\,\|f\|^2}_{\ge -x^*C^*Cx \text{ by } (*)} + x^*(A^*P + PA)x \\
&\ge \|r\,B^*Px + r^{-1}\,f\|_2^2 \underbrace{-r^2\,x^*PBB^*Px - x^*C^*Cx + x^*(A^*P + PA)x}_{>0 \text{ by the first part of the proof}}
\end{aligned}$$

Summary: We have along the solutions of the ODE,

$$x^*Px < 0, \qquad \frac{d}{dt}x^*Px > 0, \qquad \text{if } x \ne 0.$$

This implies

$$\lim_{t\to\infty} x(t) = 0.$$

Formula for the real stability radius $r_{\mathbb{R}}(A,B,C)$

In the theorem below $\sigma_2(M)$ denotes the second largest singular value of M.

Theorem. If $\lambda \in \mathbb{C}$ is not an eigenvalue of A then there exists a **real** matrix Δ with

$$\|\Delta\|_2 = \left(\inf_{\gamma\in(0,1]} \sigma_2 \left(\begin{bmatrix} \Re(G(\lambda)) & -\gamma^{-1}\Im(G(\lambda)) \\ \gamma\,\Im(G(\lambda)) & \Re(G(\lambda)) \end{bmatrix} \right) \right)^{-1}$$

such that λ is an eigenvalue of $A_\Delta = A + B\Delta C$.
There is no matrix of smaller norm with the latter property.

Consequence for the **real** stability radius of stable A:

$$\begin{aligned} r_{\mathbb{R}}(A,B,C) &= \inf\{\rho > 0 \mid \exists \text{real } \Delta : (\|\Delta\|_2 = \rho \text{ and } A_\Delta \text{ has eigenvalue on imaginary axis})\} \\ &= \frac{1}{\sup_{\omega\in\mathbb{R}} \inf_{\gamma\in(0,1]} \sigma_2 \left(\begin{bmatrix} \Re(G(\lambda)) & -\gamma^{-1}\Im(G(\lambda)) \\ \gamma\,\Im(G(\lambda)) & \Re(G(\lambda)) \end{bmatrix} \right)}. \end{aligned}$$

Unfortunately:

For nonlinear real feedback the same statement holds as for nonlinear complex feedback.
I.e. the complex stability radius and **not** the real applies for real nonlinear feedback.

Literature

(1) Book: Hinrichsen, Pritchard. Mathematical Systems Theory 1.

(2) Qiu, Bernhardson, Rantzer, Davison, Young, Doyle.
A formula for computation of the real stability radius.
Automatica 31,6 (1995)

(3) Bernhardson, Rantzer, Qiu.
Real perturbation values and real quadratic forms
in a complex vector space.
Linear Alg. Appl. 270 (1998)

(4) Genin, Stefan, van Dooren.
Real and complex stability radii of polynomial matrices.
Linear Alg. Appl. 351-352 (2002)

Perturbing Matrix polynomials

Perturbed companion form of monic matrix polynomial:

$$\begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ -P_0-\Delta_0 & \dots & \dots & -P_{n-1}-\Delta_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ -P_0 & \dots & \dots & -P_{n-1} \end{bmatrix} - \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}}_{C} \begin{bmatrix} \Delta_0 & \dots & \Delta_{n-1} \end{bmatrix}$$