# GIAN course: Singular optimal control

## Slide collection 6

## Controllability, Pole placement, Stabilization

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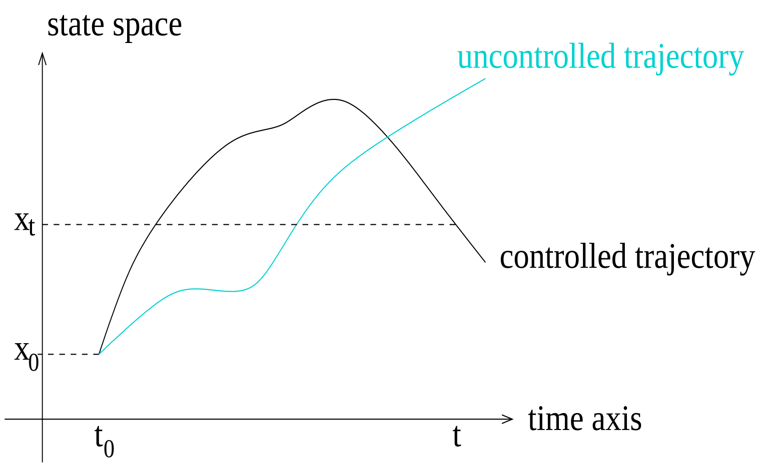
## Definition of controllability

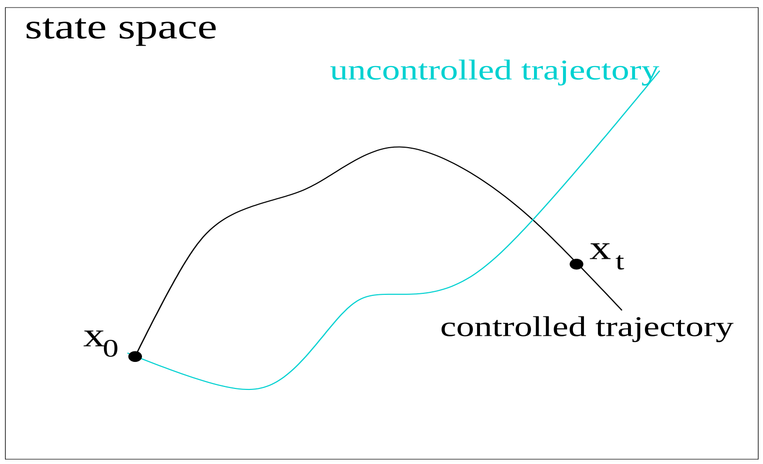
The ODE/DAE

$$E(s)\,\dot{x}(s) = A(s)\,x(s) + B(s)\,u(s) \qquad (*)$$

is said to be controllable from state $x_0$ to the state $x_t$ in the time interval $[t_0, t]$ if there exists a control $u$ s.t. $(*)$ has a solution satisfying the boundary conditions

$$x(t_0) = x_0 \quad \text{and} \quad x(t) = x_t.$$




On the following pages we first treat the ODE case and subsequently the DAE case with constant coefficients.

## Preliminary: How to solve an underdetermined linear equation

For a nonsingular square matrix $A \in \mathbb{C}^{n\times n}$: $\quad Au = z \;\Rightarrow\; u = A^{-1}z$

For a matrix $A \in \mathbb{C}^{m\times n}$ with $m < n$ (less rows than columns)

the equation $Au = z$ has either none or finitely many solutions $u$.

If the square matrix $AA^* \in \mathbb{C}^{m\times m}$ is nonsingular then one solution is given as

$$u_0 = A^* \underbrace{(AA^*)^{-1} z}_{=:\nu} \quad \Rightarrow \quad Au_0 = z.$$

If we define $W := AA^*$ then $u_0$ is computed in the following steps:

Step 1: Solve $W\nu = z$ for $\nu$. Step 2: Set $u_0 = A^*\nu$.

Note: the equation in Step 1 may have a solution even if $W$ is singular.
In this case $u_0$ also satisfies $Au_0 = z$.

Structure of $W$: Let $a_j^*$ denote the $j$th row of $A$. Then

$$A = \begin{bmatrix} a_1^* \\ \vdots \\ a_m^* \end{bmatrix} \qquad \Rightarrow \qquad W = AA^* = \begin{bmatrix} a_1^* a_1 & \dots & a_1^* a_m \\ \vdots & & \vdots \\ a_m^* a_1 & \dots & a_m^* a_m \end{bmatrix}.$$

Thus, $W$ is matrix of inner products. Such a matrix is called **Gramian matrix**.
$W$ is always Hermitian and positive semidefinite.

What is special about $u_0$: among all solution $u$ of $Au = z$

the vector $u_0$ has smallest norm: $\quad Au = z \quad \Rightarrow \quad \|u\|_2^2 \ge \|u_0\|_2^2 = z^*\nu = z^*W^{-1}z.$

## Preliminary continued

Result from slide before:
If the square matrix $W = AA^* \in \mathbb{C}^{m\times m}$ is nonsingular then we have

$$u_0 = A^* \underbrace{(AA^*)^{-1} z}_{=:\nu} \quad \Rightarrow \quad Au_0 = z.$$

This can be modified. Let $R \in \mathbb{C}^{m\times m}$ be a positive definite matrix. Redefine $u_0$ as

$$u_0 = RA^* \underbrace{(ARA^*)^{-1} z}_{=:\nu} \quad \Rightarrow \quad Au_0 = z.$$

The associated Gramian then is $W_R := ARA^*$. Solution procedure:

Step 1: Solve $W_R\, \nu = z$ for $\nu$. Step 2: set $u_0 = RA^*\nu$.

**Lemma.** Among all solutions of $Au = z$ the solution $u_0$ minimizes the norm $\|u\|_{R^{-1}} := \sqrt{u^* R^{-1} u}$, i.e.

$$\|u\|^2_{R^{-1}} \geq \|u_0\|^2_{R^{-1}} = z^*\nu = z^* W_R^{-1} z.$$

**Proof.** Any solution $u$ of $Au = z$ can be written as $u = u_0 + h$, where $Ah = 0$. The latter implies $h^* R^{-1} u_0 = h^* R^{-1} R A^* \nu = h^* A^* \nu = (Ah)^* \nu = 0$. Thus,

$$\begin{aligned} \|u\|^2_{R^{-1}} &= (u_0 + h) R^{-1} (u_0 + h) \\ &= u_0^* R^{-1} u_0 + h^* R^{-1} h + 2\, \underbrace{\Re(h^* R^{-1} u_0)}_{=0} \\ &\geq u_0^* R^{-1} u_0 = u_0^* A^* \nu = z^* \nu. \end{aligned}$$

## Preliminary continued

We have shown: If $W_R\nu = z$ has a solution $\nu$ then $Au = z$ has a solution, namely $u = u_0 = RA^*\nu$.

Question: Is it possible that $Au = z$ has a solution but $W_R\nu = z$ has no solution?

Answer: No, this is not possible. We have range $W_R$ = range $A$.

**Reason:** $(\text{range}\, W_R)^\perp \subseteq (\text{range}\, A)^\perp$.

Proof.

$$\begin{aligned} h \in (\text{range}\, W_R)^\perp \quad &\Rightarrow \quad 0 = h^* W_R \nu \quad \text{for all } \nu \\ &\Rightarrow \quad 0 = h^* W_R h = h^* A R A^* h = (h^* A) R (h^* A)^* \\ &\Rightarrow \quad h^* A = 0 \qquad \text{(since } R \text{ is positive definite)} \\ &\Rightarrow \quad h \in (\text{range}\, A)^\perp. \end{aligned}$$

## All these considerations with matrices $A$ carry over to linear operators

$$\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2, \qquad \dim \mathcal{H}_1 \geq \dim \mathcal{H}_2, \qquad \text{range}\, \mathcal{A} \text{ closed.}$$

**Further definition:** If $W_R\nu = z$ has a solution. Then there is also a solution $\nu_0$ of smallest 2-norm. $\nu_0$ depends linearly on $z$. Thus, there is a matrix $W_R^\dagger$ s.t. $\nu_0 := W_R^\dagger z$ if there exists a solution and $W_R^\dagger z := 0$ if $z \in (\text{range}\, W_R)^\perp$. $W_R^\dagger$ is called the **Moore-Penrose generalized inverse** of $W_R$.

## The controllable subspace

The solution of the initial value problem

$$x(t_0) = x_0, \quad \dot{x}(s) = A(s)\,x(s) + B(s)\,u(s) \qquad (*)$$

at time $t$ is

$$x(t) = \Phi(t, t_0)\,x_0 + \mathcal{A}(u), \qquad \mathcal{A}(u) := \int_{t_0}^{t} \Phi(t, s)\,B(s)\,u(s)\,ds,$$

where $\Phi(\cdot, \cdot)$ is the transition matrix satisfying

$$\frac{d}{dt}\Phi(t, t_0) = A(t)\,\Phi(t, t_0), \qquad \Phi(t_0, t_0) = I.$$

The **controllable subspace** for the time interval $[t_0, t]$ associated with $(A, B)$ is the range of $\mathcal{A}$,

$$\mathcal{C}(A, B, t_0, t) \;\; := \;\; \{\, z \in \mathbb{C}^n \mid \exists u \in \mathcal{L}_2([t_0, t]) : \; z = \mathcal{A}(u) \,\}.$$

The definitions imply that

$x_0$ is controllable to $x_t$ in the time interval $[t_0, t]$ iff $x_t - \Phi(t, t_0)\,x_0 \in \mathcal{C}(A, B, t_0, t)$.

## Controllability gramian

The **controllability gramian** of the linear system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$

with respect to the weight function $R(s) \in \mathbb{C}^{n\times n}$ and the interval $[t_0, t]$ is defined as

$$W_R(A, B, t_0, t) := \int_{t_0}^{t} \Phi(t,s)B(s)R(s)B(s)^*\Phi(t,s)^*\,ds, \qquad t \geq t_0,$$

where $\Phi(\cdot,\cdot)$ is the transition matrix of the ODE $\dot{x} = A\,x$, $(\frac{d}{dt}\Phi(t,t_0) = A(t)\,\Phi(t,t_0))$, and the weight $R(s)$ is Hermitian and positive definite almost everywhere.

**Remarks.**

- If $A, B, R$ are constant then $W_R(A, B, t_0, t) = \displaystyle\int_{t_0}^{t} e^{A(t-s)}B\,R\,B^* e^{A^*(t-s)}\,ds.$
- Note: $W_R(A, B, t_0, t)$ is Hermitian and positive semidefinite. Moreover,
$$t_0 \leq t \leq t' \qquad \Rightarrow \qquad 0 \leq W_R(A,B,t_0,t) \leq W_R(A,B,t_0,t').$$
- The function $W(t) := W_R(A, B, t_0, t)$ fulfills the Lyapunov ODE
$$\dot{W} = A\,W + W\,A^* + BRB^*.$$
- If $W$ is nonsingular then $P := W^{-1}$ satisfies the Riccati ODE
$$-\dot{P} = P\,A + A^*\,P + P\,BRB^*P.$$

## Fundamental fact on controllability of ODE.

**Main Proposition on controllability.** With the definitions from the slide before:

(i) For $\nu \in \mathbb{R}^n$ and $z \in \mathrm{Range}(\, W_R(A,B,t_0,t)\,)$,

$$W_R(A,B,t_0,t)\,\nu = z \quad \Rightarrow \quad \begin{cases} \displaystyle\int_{t_0}^{t} \Phi(t,s)\,B(s)\,u_0(s)\,ds = z, \\ \qquad\qquad \text{where } u_0(s) := R(s)\,B(t)^*\,\Phi(t,s)^*\,\nu. \end{cases} \qquad (*)$$

The vector $\nu$ can be chosen as $\nu = W_R(A,B,t_0,t)^\dagger\, z$ (Penrose inverse).

(ii) $\mathcal{C}(A,B,t_0,t) = \mathrm{Range}(\, W_R(A,B,t_0,t)\,)$ $\qquad (**)$

(iii) $\mathcal{C}(A,B,t_0,t)^\perp = \mathrm{Ker}(\, W_R(A,B,t_0,t)\,) = \{v \in \mathbb{C}^n \mid v^*\Phi(t,s)\,B(s) = 0, s \in [t_0,t]\,\}$

(iv) Among all controls $u$ with $\int_{t_0}^{t} \Phi(t,s)\,B(s)\,u(s)\,ds = z$ the control $u_0$ in $(**)$ minimizes the integral

$$J(u) = \int_{t_0}^{t} u(s)^* R(s)^{-1} u(s)\,ds.$$

The minimum is

$$J(u_0) = z^*\nu = z^*\,W_R(A,B,t_0,t)^\dagger\, z.$$

(v) The control $u_0$ can be written as

$$u_0(s) = R(s)\,B(s)^*\lambda(s), \qquad \lambda(t) = \nu, \quad \dot\lambda(s) = -A(s)^*\,\lambda(s) \quad \text{(adjoint ODE)}.$$

## Proof of the main proposition.

The implication $(*)$ is trivial. It implies Range$(\, W_R(A,B,t_0,t)\,) \subseteq \mathcal{C}(A,B,t_0,t)$.

The opposite inclusion is shown via orthogonal complements:

$$
\begin{aligned}
v \in \text{Range}(W_R(A,B,t_0,t))^\perp \quad \Rightarrow \quad 0 &= v^* \int_{t_0}^t \Phi(t,s)\, B(s)\, R(s)\, B(s)^* \Phi(t,s)^*\, ds\; v \\
&= \int_{t_0}^t w(s)^* R(s)\, w(s)\, ds \qquad w(s)^* := v^* \Phi(t,s)\, B(s). \\
\Rightarrow \quad 0 &= w(s)^* R(s)\, w(s) \quad \text{almost everywhere} \\
\Rightarrow \quad 0 &= w(s) \quad \text{almost everywhere} \\
\Rightarrow \quad 0 &= v^* \int_{t_0}^t \Phi(t,s)\, B(s)\, u(s)\, ds \quad \forall\, u. \\
\Rightarrow \quad & v \in \mathcal{C}(A,B,t_0,t)^\perp .
\end{aligned}
$$

Thus, $\mathcal{C}(A,B,t_0,t) \subseteq$ Range$(W(A,B,t_0,t))$. Thus, (ii) and (iii).
For any control $u(s) = u_0(s) + h(s)$

with $z = \int_{t_0}^t \Phi(t,s)\, B(s)\, u(s)\, ds$, we have $0 = \int_{t_0}^t \Phi(t,s)\, B(s)\, h(s)\, ds$.

Thus, $0 = \int_{t_0}^t z^* \Phi(t,s)\, B(s)\, R(s)\, R(s)^{-1}\, h(s)\, ds = \int_{t_0}^t u_0(s)^* R(s)^{-1}\, h(s)\, ds$, and

$$J(u) = \int_{t_0}^t (u_0(s) + h(s))^*\, R(s)^{-1}\, (u_0(s) + h(s))\, ds \;=\; J(u_0) + J(h) \geq J(u_0) \qquad \Rightarrow (iii).$$

(iv) holds, since $\lambda(s) = \Phi(t,s)^* \nu$ and $\frac{d}{ds}\Phi(t,s) = -\Phi(t,s)\, A(s)$.

## Fundamental fact on controllability of ODE (again).

**Corollary.** The following statements are equivalent.

(1) There exists a control $u$ such that

$$x(t_0) = x_0, \quad \dot{x}(s) = A(s)\,x(s) + B(s)\,u(s), \quad x(t) = x_t \qquad (*)$$

(2) There exists a solution $\nu \in \mathbb{C}^n$ of the linear equation

$$W_R(A, B, t_0, t)\,\nu = x_t - \Phi(t, t_0)\,x_0. \qquad (**)$$

If (2) holds then $(*)$ is fulfilled for the control

$$u_0(s) = R(s)\,B(s)^*\,\Phi(t,s)^*\nu = R(s)\,B(s)^*\lambda(s), \quad \lambda(t) = \nu, \quad \dot{\lambda}(s) = -A(s)^*\,\lambda(s).$$

Furthermore, among all controls satisfying $(*)$ the control $u_0$ minimizes the integral

$$J(u) = \int_{t_0}^{t} u(s)^*\,R(s)^{-1}\,u(s)\,ds.$$

The minimum is

$$J(u_0) = z^*\nu = z^*\,W_R(A, B, t_0, t)^\dagger\,z.$$

## Output control.

We consider a linear system with output $y = C\,x$, where $C \in \mathbb{C}^{p\times n}$ is a constant matrix. The statement below is shown analogously to the main proposition.

**Proposition.** The following statements are equivalent.

(1) There exists a control $u$ such that

$$x(t_0) = x_0, \quad \dot{x}(s) = A(s)\,x(s) + B(s)\,u(s), \quad C\,x(t) = c_t \qquad (*)$$

(2) There exists a solution $\nu \in \mathbb{C}^n$ of the linear equation

$$C\,W_R(A, B, t_0, t)\,C^*\,\nu = c_t - C\Phi(t, t_0)\,x_0. \qquad (**)$$

If (2) holds then $(*)$ is fulfilled for the control

$$u_0(s) = R(s)\,B(s)^*\,\Phi(t, s)^* C^* \nu = R(s)\,B(s)^* \lambda(s), \quad \lambda(t) = \nu, \quad \dot{\lambda}(s) = -A(s)^* C^*\,\lambda(s).$$

Furthermore, among all controls satisfying $(*)$ the control $u_0$ minimizes the integral

$$J(u) = \int_{t_0}^{t} u(s)^*\,R(s)^{-1}\,u(s)\,ds.$$

The minimum is

$$J(u_0) = z^*\nu = z^*\,(C\,W_R(A, B, t_0, t)C^*)^\dagger\,z.$$

## The time invariant case and the Kalman matrix

**Proposition.** If $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times p}$ are constant then for $t > t_0$,

$$\mathcal{C}(A,B,t_0,t) = \text{Range}(\,\mathcal{K}(A,B)\,) =: \mathcal{C}(A,B)$$

where

$$\mathcal{K}(A,B) = [B \;\; AB \;\; A^2B \;\; \ldots \;\; A^{n-1}B] \in \mathbb{C}^{n\times np} \qquad \text{(Kalman matrix).}$$

**Proof.** We have

$$\int_{t_0}^{t} e^{A(t-s)} B\,u(s)ds = \int_{t_0}^{t} \left( \sum_{k=0}^{n-1} \phi_k(t-s) A^k \right) B\,u(s)\,ds = [B \;\; AB \;\; \ldots \;\; A^{n-1}B] \begin{bmatrix} \int \phi_0(t-s)\,u(s)\,ds \\ \int \phi_1(t-s)\,u(s)\,ds \\ \vdots \\ \int \phi_{n-1}(t-s)\,u(s)\,ds \end{bmatrix},$$

where $\chi_A(\frac{d}{dt})\phi_k = 0$, $\phi_k^{(j)}(0) = \delta_{j,k}$. Thus, $\mathcal{C}(A,B,t_0,t) \subseteq \text{Range}(\,\mathcal{K}(A,B)\,)$.

The opposite inclusion is shown via orthogonal complements:

$$\begin{aligned} v \in \mathcal{C}(A,B,t_0,t)^{\perp} \quad &\Rightarrow \quad 0 = \int_{t_0}^{t} v^*\, e^{A(t-s)} B\,u(s)ds \quad \forall u \\ &\Rightarrow \quad 0 \equiv v^*\, e^{A(t-s)} B \\ &\Rightarrow \quad 0 = \left.\frac{d^k}{ds^k}\right|_{s=t} v^*\, e^{A(t-s)} B = \pm v^* A^k B \quad \Rightarrow \quad v \in \text{Range}(\,\mathcal{K}(A,B)\,)^{\perp}. \end{aligned}$$

**Remark.** By Cayley-Hamilton, $\mathcal{C}(A,B)$ contains Range$(A^kB)$ for all $k \geq 0$.
Thus, $\mathcal{C}(A,B)$ is the smallest $A$-invariant subspace containing Range$(B)$.

## Kalman decomposition

Let $V = [V_1, V_2] \in \mathbb{C}^n$ be a nonsingular matrix such that the columns of $V_1$ form a basis of $\mathcal{C}(A,B) = \text{range}(\mathcal{K}(A,B))$. Then, since $\mathcal{C}(A,B)$ is an $A$-invariant subspace which contains range($B$),

$$AV_k = V_k\, A_{11}, \qquad B = V_1\, B_1, \qquad \text{for some matrices } A_{11}, B_1.$$

Thus,

$$A = V \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} V^{-1}, \qquad B = V \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

For $x = V \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we have

$$\dot{x} = Ax + Bu \quad \Leftrightarrow \quad \begin{cases} \dot{x}_1 = A_{11}\, x_1 + A_{12}\, x_2 + B_1\, u, \\ \dot{x}_2 = A_{22}\, x_2. \end{cases}$$

$x_2$ is completely uncontrollable.
The eigenvalues of $A$, which are also eigenvalues of $A_{22}$ are called **uncontrollable modes.**

**Hautus Test.** For any eigenvalue $\lambda$ of $A$: $\ker(A-\lambda I)^n \subseteq \mathcal{C}(A,B) \Leftrightarrow \operatorname{rank}[A-\lambda I, B] = n$.

**Corollary.** $\mathcal{C}(A,B) = \mathbb{C}^n$ (complete controllability) $\Leftrightarrow$ $\operatorname{rank}[A-\lambda I, B] = n$ for all $\lambda \in \mathbb{C}$.

**Proof.**$\Leftarrow$. Since $[A-\lambda I, B]$ has full row rank the following sequence has solutions $z_k, u_k$ for any initial vector $y$:

$$
\begin{aligned}
y &= (A-\lambda I)z_1 + B\,u_0 \\
z_1 &= (A-\lambda I)z_2 + B\,u_1 \\
z_2 &= (A-\lambda I)z_3 + B\,u_2 \\
&\vdots \\
z_{n-2} &= (A-\lambda I)z_{n-1} + B\,u_{n-2} \\
z_{n-1} &= (A-\lambda I)z_n + B\,u_{n-1}.
\end{aligned}
$$

This yields

$$y = (A-\lambda I)^n z_n + (A-\lambda I)^{n-1} B\,u_{n-1} + \ldots + (A-\lambda I)\,B\,u_1 + B\,u_0.$$

Let $\pi_\lambda$ be the spectral projector onto $\ker(A-\lambda I)^n$.

Since $\pi_\lambda$ is a polynomial of $A$ we have

$$\pi_\lambda(A-\lambda I)^n = (A-\lambda I)^n \pi_\lambda = 0,$$

and for $y \in \ker(A-\lambda I)^n$,

$$y = \pi_\lambda\, y = \pi_\lambda((A-\lambda I)^{n-1} B\,u_{n-1} + \ldots + (A-\lambda I)\,B\,u_1 + B\,u_0) \in \mathcal{C}(A,B).$$

Other direction: see next slide.

**Proof continued**

Let $V = [V_1, V_2, \ldots, V_r]$ where $V_k$ is a matrix whose columns form a basis of $\mathrm{Ker}\,(A - \lambda_k I)^{m_k}$. Then there are nilpotent matrices $\widehat{N}_k$ such that

$$A\,V_k = V_k\,(\lambda_k I + \widehat{N}_k).$$

Thus,

$$A \;=\; V \begin{bmatrix} \lambda_1 I + \widehat{N}_1 & & & \\ & \lambda_2 I + \widehat{N}_2 & & \\ & & \ddots & \\ & & & \lambda_r I + \widehat{N}_r \end{bmatrix} V^{-1}, \qquad B = V \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_r \end{bmatrix} \text{ for some } B_k.$$

It follows that

$$\begin{aligned} &\mathrm{Ker}\,(A - \lambda_k I)^{m_k} \subseteq \mathrm{range}\,[B\ AB \ldots\ A^{n-1}B\,] \\ &\Leftrightarrow\ \underbrace{[B_k\ (\lambda_k I + \widehat{N}_k)B_k\ \ \ldots\ (\lambda_k I + \widehat{N}_k)^{n-1}B_k]}_{K} \text{ has full rank,} \end{aligned}$$

and

$$\mathrm{rank}\,[A - \lambda I,\ B] = \sum_k \mathrm{rank}\,[\lambda_k I + \widehat{N}_k - \lambda I,\ B_k].$$

Thus $\mathrm{rank}\,[A - \lambda I,\ B] < n$ if $\lambda = \lambda_k$ for some $k$ and $[\widehat{N}_k,\ B_k]$ has not full rank.

In the latter case there is a vector $y$ such that $0 = y^*[\widehat{N}_k,\ B_k] = [y^*\widehat{N}_k, y^*\,B_k]$, which implies $0 = y^*K. \Rightarrow K$ has not full rank.

## An observation on the Kalman matrix

$\chi_A(\lambda) = \lambda^n + \sum_k p_k \lambda^k$ be the characteristic polynomial of $A$.

From the Cayley-Hamilton theorem, $0 = \chi_A(A) = A^n + \sum_{k=0}^{n-1} p_k A^k$, it follows that

$A^n B = -\sum_k p_k A^k B. \quad \Rightarrow$

$$A \underbrace{[B \ \ AB \ \ A^2B \ \ \dots \ \ A^{n-1}B]}_{\mathcal{K}(A,B)} = [B \ \ AB \ \ A^2B \ \ \dots \ \ A^{n-1}B] \underbrace{\begin{bmatrix} 0 & & & -p_0 I \\ I & & & -p_1 I \\ & \ddots & & \vdots \\ & & I & -p_{n-1} I \end{bmatrix}}_{=:C_\sharp}$$

This implies

(1) $A^j \mathcal{K}(A,B) = \mathcal{K}(A,B) C_\sharp^j, \quad j = 0, 1, 2 \dots,$

(2) $e^{At} \mathcal{K}(A,B) = \mathcal{K}(A,B) e^{C_\sharp t}$.

(3) If $b \in \mathbb{C}^n$ and $(A, b)$ is completely controllable,

$$C_\sharp = \mathcal{K}(A,b)^{-1} A \mathcal{K}(A,b), \qquad \mathcal{K}(A,b)^{-1} b = e_1$$

In this case

$$\dot{x} = Ax + Bu \quad \Leftrightarrow \quad \left( x = \mathcal{K}(A,b) x_r \text{ and } \dot{x}_r = C_\sharp x_r + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \right)$$

**Another control construction.** Let

$$z = B\,z_0 + AB\,z_1 + \cdots + A^{n-1}B\,z_{n-1} \in \text{Range}(\,\mathcal{K}(A,B)\,).$$

Let

$$\psi : [t_0, t] \to \mathbb{R} \;\text{ smooth}, \quad \int_{t_0}^{t} \psi(s)\,ds = 1, \quad \psi^{(k)}(t_0) = \psi^{(k)}(t) = 0, \text{ for } k = 0, \ldots, n-1.$$

Let

$$u(s) = \sum_{k=0}^{n-1} u_k^{(k)}(s), \quad \text{where} \quad \begin{bmatrix} u_0(s) \\ \vdots \\ u_{n-1}(s) \end{bmatrix} = e^{-C_\sharp(t-s)} \begin{bmatrix} z_0 \\ \vdots \\ z_{n-1} \end{bmatrix} \psi(s), \quad C_\sharp = \begin{bmatrix} 0 & & & -p_0\,I \\ I & & & -p_1\,I \\ & \ddots & & \\ & & I & -p_{n-1}\,I \end{bmatrix},$$

where $\chi_A(\lambda) = \lambda^n + \sum_k p_k\,\lambda^k$ is the characteristic polynomial of $A$. Then

$$\int_{t_0}^{t} e^{A(t-s)} B\,u(s)\,ds = z.$$

**Proof.** Using partial integration (boundary terms vanish) and the observation from the former page we have

$$\int_{t_0}^{t} e^{A(t-s)} B\,u(s)\,ds \;=\; \int_{t_0}^{t} e^{A(t-s)} B \sum_{k=0}^{n-1} u_k^{(k)}(s)\,ds \;=\; \int_{t_0}^{t} e^{A(t-s)} \sum_{k=0}^{n-1} A^k B\,u_k(s)\,ds$$

$$= \int_{t_0}^{t} e^{A(t-s)} \mathcal{K}(A,B) \begin{bmatrix} u_0(s) \\ \vdots \\ u_{n-1}(s) \end{bmatrix} ds \;=\; \int_{t_0}^{t} \mathcal{K}(A,B) e^{C_\sharp(t-s)} \begin{bmatrix} u_0(s) \\ \vdots \\ u_{n-1}(s) \end{bmatrix} ds \;=\; \int_{t_0}^{t} \mathcal{K}(A,B) \begin{bmatrix} z_0 \\ \vdots \\ z_{n-1} \end{bmatrix} \psi(s)\,ds = z$$

## Controllability of polynomial equations

The linear system associated with the scalar polynomial ODE

$$x^{(n)} + p_{n-1}\, x^{(n-1)} + \ldots + p_1\, \dot{x} + p_0\, x = f$$

is completely controllable.

**Proof.** Use the second companion form

$$\frac{d}{dt}\begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & & & -p_0 \\ 1 & & & -p_1 \\ & \ddots & & \\ & & 1 & -p_{n-1} \end{bmatrix}}_{C_\sharp} \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{b} f$$

We have $C_\sharp b = C_\sharp\, e_1 = e_2$, $C_\sharp^2\, b = C_\sharp\, e_2 = e_3$ etc.. Thus,

$$\mathcal{K}(C_\sharp, b) = I.$$

## Similarity to companion matrix

**Proposition.** Let $A \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^n$. Suppose that $\mathcal{K} = [b \;\; Ab \; A^2b \;\; \ldots A^{n-1}b]$ is nonsingular (i.e. $(A,b)$ controllable). Let $\chi_A(x) = x^n + p_{n-1}\,x^{n-1} + \ldots + p_0$ be the characteristic polynomial of $A$. Let

$$C_\sharp = \begin{bmatrix} 0 & & & -p_0 \\ 1 & & & -p_1 \\ & \ddots & & \vdots \\ & & 1 & -p_{n-1} \end{bmatrix}, \qquad C_\flat = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -p_0 & -p_1 & \ldots & -p_{n-1} \end{bmatrix}.$$

Then

$$A = \mathcal{K}\,C_\sharp\,\mathcal{K}^{-1} = S^{-1}\,C_\flat\,S, \qquad \text{and} \quad \mathcal{K}^{-1}b = e_1, \;\; Sb = e_n.$$

where $e_k$ it the $k$-th canonical basis vector and

$$S := \begin{bmatrix} s^\top \\ s^\top A \\ \vdots \\ s^\top A^{n-1} \end{bmatrix}, \qquad s^\top = \text{lower row of } \mathcal{K}^{-1} = e_n^\top \mathcal{K}^{-1}.$$

Proof. By Cayley-Hamilton: $A^n = -\sum_{k=0}^{n-1} p_k\,A^k$. Thus, $A\,\mathcal{K} = \mathcal{K}\,C_a^\sharp$ and $S\,A = C_a^\flat S$. The second relation does not depend on the choice of $s$, but the choice above implies invertibility of $S$. Obviously, $\mathcal{K}e_1 = b$. In general $\mathcal{K}e_k = A^{k-1}b$. $\Rightarrow$ $\delta_{nk} = e_n^\top e_k = e_n^\top \mathcal{K}^{-1}A^{k-1}b = s^\top A^{k-1}b \;\; \Rightarrow \;\; Sb = e_n$.

**Remark.** Similarity of $A$ to a companion matrix implies that $A$ has only one Jordan block to each eigenvalue. This follows from the Jordan canonical form of companion matrices. Thus if $A$ has multiple Jordan blocks to the same eigenvalue then $(A,b)$ is not controllable for any $b$.

## Pole shifting with single input, controllable canonical form

A single input system

$$\dot{x} = Ax + b\,u, \qquad u = u(t) \in \mathbb{C}$$

is said to be in **controllable canonical form** if

$$A = C_\flat = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -p_0 & -p_1 & \dots & -p_{n-1} \end{bmatrix}, \qquad b = e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

**Reminder.**

Linear system: $\dot{x} = Ax + Bu$. Feedback: $u = -Fx$. closed loop: $\dot{x} = (A - BF)x$

Eigenvalues of $A =$ Poles of $(\lambda I - A)^{-1}$.

Since the coefficients of the characteristic polynomial of a companion matrix $C_\flat$ can be found in its last row, we can change that polynomial arbitrarily via feedback as follows. For $(A, b)$ in controllable canonical form:

Let $F = f^\top = [p_0 - q_0 \quad p_1 - q_1 \quad \dots \quad p_{n-1} - q_{n-1}]$. Then

$$A_q := A - b\,f^\top = C_\flat - e_n\,f^\top = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -q_0 & -q_1 & \dots & -q_{n-1} \end{bmatrix},$$

# Pole shifting with single input, Ackermann formula

**Reminder.**

Linear system: $\dot{x} = Ax + Bu$. Feedback: $u = -Fx$. closed loop: $\dot{x} = (A - BF)x$

Eigenvalues of $A =$ Poles of $(\lambda I - A)^{-1}$.

**Theorem.** Suppose $(A, b)$ is controllable. Let $s^\top$ be the lower row of $\mathcal{K}^{-1}$, where $\mathcal{K} = [b \;\; Ab \; A^2b \;\; \ldots A^{n-1}b]$. Let $q$ be a monic polynomial and let

$$A_q = A - b\,f^\top, \qquad \text{with } f^\top = s^\top q(A). \qquad \textbf{(Ackermann formula)}$$

Then $A_q$ is similar to

$$C_\flat = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -q_0 & -q_1 & \ldots & -q_{n-1} \end{bmatrix}.$$

Thus $q$ is the characteristic polynomial of $A_q$. Eigenvalues are zeros of $q$.

<u>Proof.</u> $A_q$ does not change if $q$ is replaced by $q - \chi_A$: $A_q = A - b\,s^\top(q(A) - \chi_A(A))$.
Now, apply similarity transformation with the matrix $S$ from the former slide.

## Reduction to controllability by single input via feedback

**Lemma (Heymann).**
Let $(A, B)$ be completely controllable. Let $0 \neq b \in \text{range}(B)$.
Then there exists a feedback matrix $F$ such that $(A + BF, b)$ is controllable.

Combining this with Ackermann's formula we have

**Corollary (Pole placement).**
If $(A, B)$ is controllable then to any $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ there is a feedback $\tilde{F}$ such that the $\lambda_k$ are the eigenvalues of $A + B\tilde{F}$.
(If all matrices are real, the nonreal $\lambda_k$ have to be given in conjugate complex pairs).

**Proof sketch for the lemma.** Basic idea: construct basis $X = [x_1\ x_2\ \ldots\ x_n]$ of $\mathbb{C}^n$ s.t.

$$x_1 = b, \quad x_{k+1} = A\,x_k + B\,u_k \quad \text{for some } u_k,\ k \leq n-1.$$

Existence of such a basis follows from controllability. Let $u_n$ be arbitrary, and define

$$F := [u_1\ u_2\ \ldots\ u_n]\,X^{-1} \quad \Rightarrow \quad Fx_k = u_k \quad \Rightarrow \quad (A + BF)x_k = x_{k+1}, \quad k \leq n-1.$$

Thus, the Kalman matrix

$$[b \quad (A + BF)b \quad (A + BF)^2 b \quad \ldots \quad (A + BF)^{n-1} b] = [x_1\ x_2\ x_3\ \ldots\ x_n]$$

has full rank.

# A feedback stabilization method (simple version)

**Proposition (feedback stabilization method of Bass).**

Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$. Suppose that $(A,B)$ is controllable.

Let $\alpha > \max\{\,|\lambda| \mid \lambda$ Eigenvalue of $A\}$. Then the Lyapunov equation

$$(A+\alpha\, I)P + P(A^* + \alpha\, I) = 2\, BRB^* \qquad (*)$$

has a unique solution $P$. This solution is Hermitian and positive definite.

The matrix $A - B\underbrace{RB^*P^{-1}}_{F}$ is stable.

<u>Proof.</u> $(*)$ is equivalent to

$$(-A-\alpha\, I)P + P(-A^* - \alpha\, I) = -(\sqrt{2}\, B)R(\sqrt{2}\, B)^*.$$

Controllability of $(A,B)$ implies controllability of $(-A-\alpha\, I, \sqrt{2}\, B)$. The choice of $\alpha$ yields that $-A-\alpha\, I$ is stable. Hence, by the results from the former slides,

$$P = 2\int_0^\infty e^{(-A-\alpha\, I)\,t} BRB^* e^{(-A^*-\alpha\, I)\,t}\, dt$$

solves the Lyapunov equation. $P$ is pos. def. by controllability. The integral is already pos. def. if $\infty$ is replaced by a finite time (controllability gramian of controllable pair). Now $(*)$ can be rearranged to

$$(A - BRB^*P^{-1})\, P + P\, (A - BRB^*P^{-1})^* = -2\alpha\, P.$$

It follows that $(A - BRB^*P^{-1})$ is stable because $P > 0$, $-2\alpha\, P < 0$.

## Stabilization via feedback (continued)

Consider Kalman decomposition

$$A = V \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} V^{-1}, \qquad B = V \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \qquad (A_{11}, B_1) \text{ controllable.}$$

For $x = V \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we have

$$\dot{x} = Ax + Bu \quad \Leftrightarrow \quad \begin{cases} \dot{x}_1 = A_{11}\, x_1 + A_{12}\, x_2 + B_1\, u, \\ \dot{x}_2 = A_{22}\, x_2, \end{cases}$$

If the uncontrollable matrix $A_{22}$ has no eigenvalues with nonnegative real part then $(A, B)$ can be stabilized via feedback:

Choose $\tilde{F}$ such that $A_{11} + B_1\, \tilde{F}$ is stable. Then $A + B(\tilde{F}\, V^{-1})$ is stable.

<u>By the Hautus criterion:</u>

$(A, B)$ is stabilizable if and only if $\operatorname{rank}([A - \lambda\, I, B]) = n$ for all $\lambda \in \mathbb{C}$ with $\Re(\lambda) \geq 0$.

## Pole placement with MATLAB

The commands

$$\texttt{F} = \texttt{acker}(\texttt{A}, \texttt{B}, \texttt{lambda}), \qquad \texttt{F} = \texttt{place}(\texttt{A}, \texttt{B}, \texttt{lambda})$$

return a feedback matrix $F$ such that $A - BF$ has eigenvalues specified in the vector `lambda`.

The function `acker` applies the Ackermann formula, thus $B$ must be a column vector.

The function `place` uses the algorithm given in

Kautsky, Nichols, Van Dooren: Robust Pole Assingment in linear State Feedback. International Journal of Control, 41 (1985), pp. 1129-1155.

# Controllability of descriptor systems (constant coeffcients)

We consider the controllability of regular descriptor systems in standard form

$$\underbrace{\begin{bmatrix} I & \\ & N \end{bmatrix}}_{E} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & \\ & I \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{B} u \qquad N^{i-1} \neq 0 = N^i. \qquad (*)$$

Given a time interval $[t_0, t]$ and vectors $x_{j,0}, x_{j,t}$ we seek for a control $u$ and a trajectory of $(*)$ such that $x_j(t_0) = x_{j,0}$ and $x_j(t) = x_{j,t}$ $(j = 1, 2)$.

**Recall:** the second subsystem has solution

$$x_2(s) = -\sum_{k=0}^{i-1} N^k B_2 u^{(k)}(s) \in \text{Range}(\,[B_2 \;\; NB_2 \;\; \ldots \;\; N^{i-1}B_2]\,) = \text{Range}(\,\mathcal{K}(N, B_2)\,).$$

Since it is easy to construct a $\mathcal{C}^\infty$-function with prescribed derivatives at prescribed points, we immediately have

**Proposition.** There exists a control $u = u_2$ such that $x_2(t_0) = x_{2,0}$, $x_2(t) = x_{2,t}$ iff

$$x_{2,0}, x_{2,t} \in \text{Range}(\,[B_2 \;\; NB_2 \;\; \ldots \;\; N^{i-1}B_2]\,).$$

Given such a suitable control $u_2$ further suitable controls are $u = u_2 + u_1$ where $u_1$ satisfies $u_1^{(k)}(t_0) = u_1^{(k)}(t) = 0$, $k = 0, \ldots, i-1$.

**Problem:** Find $u_1$ such that the boundary conditions of the first subsystem are matched.

# Controllability of descriptor systems (continued)

The ODE for the first subsystem is $\dot{x}_1 = A_1 x_1 + B_1(u_1 + u_2)$.
We assume that $u_2$ is fixed and seek for $u_1$ such that

$$x_1(t_0) = x_{1,0}, \quad x_1(t) = x_{1,t}, \qquad (*)$$

and

$$u_1^{(k)}(t_0) = u_1^{(k)}(t) = 0 \text{ for } k = 0, \ldots, i-1. \qquad (**)$$

For any control $u_1$ satisfying $(*)$ we have

$$x_{1,t} = e^{A_1(t-t_0)}\, x_{1,0} + \int_{t_0}^{t} e^{A_1(t-s)}\, B_1\,(u_1(s) + u_2(s))\, ds$$

This can be rewritten as

$$\int_{t_0}^{t} e^{A_1(t-s)}\, B_1\, u_1(s)\, ds = z, \qquad \text{where} \quad z := x_{1,t} - e^{A_1(t-t_0)}\, x_{1,0} - \int_{t_0}^{t} e^{A_1(t-s)}\, B_1\, u_2(s)\, ds.$$

If $z \in \mathcal{C}(A_1, B_1)$ then for any almost everywhere positive definite weight matrix $R(s)$ the control

$$u_1(s) := R(s)\, B_1^*\, e^{A_1^*(t-s)}\, \nu, \qquad \nu := W_R(A_1, B_1, t_0, t)^\dagger\, z$$

yields $(*)$. The condition $(**)$ is fulfilled if

$$R^{(k)}(t_0) = R^{(k)}(t) = 0 \text{ for } k = 0, \ldots, i-1.$$

## Rank criteria for controllability of descriptor systems

Consider the descriptor $E\dot{x} = Ax + Bu$, $A \in \mathbb{C}^{n\times n}$ with standard representation

$$\begin{bmatrix} I & \\ & N \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & \\ & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad N^{i-1} \neq 0 = N^i. \qquad (*)$$

From our considerations combined with the Hautus test the following is easily derived.

(1) The subsystem $N\dot{x}_2 = x_2 + B_2 u$ is completely controllable iff
rank$([E, B]) = n$ (equivalently: $[N, B_2]$ has full rank)

(2) The subsystem $\dot{x}_1 = A_1 x_1 + B_1 u$ is completely controllable iff
rank$([\lambda E - A, B]) = n$ for all $\lambda \in \mathbb{C}$.

(3) The full system is completely controllable iff
the conditions of (1) and (2) are both satisfied.

(4) The full system is stabilizable iff $\dot{x}_1 = A_1 x_1 + B_1 u$ is stabilizable iff
rank$([\lambda E - A, B]) = n$ for all $\lambda \in \mathbb{C}$ with $\Re(\lambda) \geq 0$.

Remark on (2): $\lambda N - I$ is nonsingular for all $\lambda$.

Remark on (4): The second subsystem is stable anyway, because any solution of the homogeneous equation $E\dot{x} = Ax$ satisfies $x_2 \equiv 0$.