GIAN course: Singular optimal control

Slide collection 5

Sylvester and Lyapunov equations
First remarks on Riccati equations

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Sylvester operators and Sylvester equations

Sylvester operator:

$$\mathcal{S} : \mathbb{K}^{n\times m} \to \mathbb{K}^{n\times m}, \qquad \mathcal{S}(X) := AX - XB, \qquad \text{where } A \in \mathbb{K}^{n\times n},\ B \in \mathbb{K}^{m\times m}.$$

Sylvester equations:

$$\underline{\text{algebraic:}}\quad AX - XB = \pm C, \qquad \underline{\text{differential:}}\quad \dot{X} = AX - XB + C.$$

Note that the square matrices A and B can be of different size.

Thus, in general X and C are not square, but of the same size.

James Joseph Sylvester (1814-1897):
Professor for mathematics and physics in UK and USA.

Main theorem on Sylvester operators

Let $\mathbb{K}$ be any field. Suppose the matrices $A \in \mathbb{K}^{n\times n}$ and $B \in \mathbb{K}^{m\times m}$ have coprime minimal polynomials (i.e. they have no common eigenvalue in an algebraicly closed extension field)

Then the **Sylvester operator**

$$\mathbb{K}^{n\times m} \ni X \longmapsto \mathcal{S}(X) = AX - XB \in \mathbb{K}^{n\times m}$$

is bijective. In other words, for any $C \in \mathbb{K}^{n\times m}$, the equation

$$AX - XB = C$$

has a unique solution. Furthermore, there are $\alpha_{jk} \in \mathbb{K}$ such that for all C:

$$X = \mathcal{S}^{-1}(C) = \sum_{jk} \alpha_{jk}\, A^j C B^k.$$

Proof. 3 proofs on the following slides.

Main theorem on Sylvester operators: first proof

Assumption: The minimal polynomials μ_A and μ_B are coprime.

We show that under this assumption the Sylvester operator

$$\mathbb{K}^{n\times m} \ni X \longmapsto \mathcal{S}(X) = AX - XB \in \mathbb{K}^{n\times m}$$

is injective. This implies that the inverse $\mathcal{S}^{-1}$ exists and there is a polynomial $g(x) = \sum_i \gamma_i \, x^i \in \mathbb{K}[x]$ such that

$$\mathcal{S}^{-1}(C) = g(\mathcal{S})(C) = \sum_i \gamma_j (AC - CB)^i = \sum_{jk} \alpha_{jk} \, A^j C B^k.$$

Proof of injectivity: Suppose that $0 = \mathcal{S}(X) = AX - XB$. Then $AX = XB$. Hence, $A^i X = X B^i$ and more generally:

$$p(A)\, X = X \, p(B) \qquad \text{for all } p \in \mathbb{K}[x].$$

In particular:

$$0 = \mu_A(A) X = X \, \mu_A(B)$$

Since μ_A and μ_B are coprime, $\mu_A(B)$ is nonsingular. Thus $X = 0$. Thus $\mathcal{S}$ is injective.

Main theorem on Sylvester operators: second proof

Observation 1: The linear operators $X \stackrel{\mathcal{A}}{\longmapsto} AX$ and $X \stackrel{\mathcal{B}}{\longmapsto} XB$ commute:

$$\mathcal{A}(\mathcal{B}X) = \mathcal{A}(XB) = AXB = \mathcal{B}(AX) = \mathcal{B}(\mathcal{A}X).$$

For commuting linear operators there is a polynomial calculus in 2 variables:

$$p(x,y) = \sum_{j,k} \alpha_{jk}\, x^j y^k \quad \longmapsto \quad p(\mathcal{A},\mathcal{B}) = \sum_{j,k} \alpha_{jk}\, \mathcal{A}^j \mathcal{B}^k.$$

The polynomial associated with the Sylvester operator

$$\mathcal{S}(X) = AX - XB = \mathcal{A}X - \mathcal{B}X = (\mathcal{A} - \mathcal{B})X$$

is $x - y$.

Observation 2: The operators $\mathcal{A}$ and $\mathcal{B}$ have the same annulating polynomials as the associated matrices:

$$0 = p(A) \quad \Leftrightarrow \quad 0 = p(A)X \ \text{ for all } X \in \mathcal{K}^{n\times n} \quad \Leftrightarrow \quad p(\mathcal{A}) = 0.$$

Analogously for $\mathcal{B}$. In particular, we have for the minimal polynomials:

$$\mu_{\mathcal{A}} = \mu_A, \qquad \mu_{\mathcal{B}} = \mu_B.$$

Main theorem on Sylvester operators: second proof

Hilbert's Weak Zero Theorem (Nullstellensatz): Suppose the polynomials $p_k(x_1, x_2, \ldots, x_r)$, $k = 1, \ldots, s$, with coefficients in $\mathbb{K}$ have no common zero in an algebraicly closed extension of $\mathbb{K}$. Then there are polynomials ϕ_k with coefficients in $\mathbb{K}$ such that

$$\sum_k \phi_k \, p_k = 1.$$

Proof: see introductory books on algebraic geometry of the blog of Terence Tao.

By assumption the minimal polynomials $\mu_{\mathcal{A}} = \mu_A$ and $\mu_{\mathcal{B}} = \mu_B$ are coprime. Thus, the polynomials in 2 variables

$$\mu_{\mathcal{A}}(x), \quad \mu_{\mathcal{B}}(y), \qquad x - y$$

have no common zero. By the weak zero theorem there are polynomials ϕ, ϕ_A, ϕ_B such that

$$1 = \phi(x, y)\,(x - y) + \phi_A(x, y)\,\mu_{\mathcal{A}}(x) + \phi_B(x, y)\,\mu_{\mathcal{B}}(y).$$

Replacing x by $\mathcal{A}$ and y by $\mathcal{B}$ yields

$$I = \phi(\mathcal{A}, \mathcal{B})\,(\mathcal{A} - \mathcal{B}).$$

Hence, $\phi(\mathcal{A}, \mathcal{B})$ is the inverse of the Sylvester operator $\mathcal{A} - \mathcal{B}$. We have

$$\mathcal{S}^{-1}(X) = \phi_0(\mathcal{A}, \mathcal{B}) = \phi(\mathcal{A}, \mathcal{B}) = \sum_{j,k} \alpha_{jk}\, \mathcal{A}^j \mathcal{B}^k X = \sum_{j,k} \alpha_{jk}\, A^j X B^k$$

for some $\alpha_{jk} \in \mathbb{K}$.

Main theorem on Sylvester operators: second proof

Actually, the Zero Theorem is not needed for the proof.

The existence of ϕ, ϕ_A, ϕ_B can shown using the Bezout identity in one variable only.

Since μ_A and μ_B are coprime, there exist polynomials $\ell_A, \ell_B \in \mathbb{K}[x]$ such that

$$\ell_A(x)\,\mu_A(x) + \ell_B(x)\,\mu_B(x) = 1.$$

Define

$$\phi(x,y) := -\ell_A(y)\,\ell_B(x)\,\frac{\mu_A(x)\mu_B(y) - \mu_A(y)\mu_B(x)}{x-y}.$$

Then $\phi(x,y)$ is a polynomial because $\mu_A(x)\mu_B(y) - \mu_A(y)\mu_B(x)$ is a sum of terms of the form $x^j y^k - y^j x^k$ and each of these terms is divisible by $x - y$. Furthermore,

$$\begin{aligned}
\phi(x,y)(x-y) &= -\ell_A(y)\ell_B(x)\,(\mu_A(x)\mu_B(y) - \mu_A(y)\mu_B(x)) \\
&= \ell_A(y)\mu_A(y)\ell_B(x)\mu_B(x) - \ell_A(y)\mu_A(x)\ell_B(x)\mu_B(y) \\
&= (1 - \ell_B(y)\mu_B(y))(1 - \ell_A(x)\mu_A(x)) - \ell_A(y)\mu_A(x)\ell_B(x)\mu_B(y) \\
&= 1 - \phi_A(x,y)\,\mu_A(x) - \phi_B(x,y)\,\mu_B(y).
\end{aligned}$$

Main theorem on Sylvester operators: third proof

Suppose B is upper triangular, $B = \begin{bmatrix} b_{11} & \dots & b_{1m} \\ & \ddots & \vdots \\ & & b_{mm} \end{bmatrix}$. Then the equation $AX - XB = C$ can be written columnwise as

$$\begin{aligned} Ax_1 - x_1\, b_{11} &= c_1, \\ Ax_2 - (x_1\, b_{12} + b_{22}\, x_2) &= c_2, \\ Ax_3 - (x_1\, b_{13} + b_{23}\, x_2 + b_{33}\, x_3) &= c_3, \\ &\vdots \end{aligned}$$

Reordering terms:

$$\begin{aligned} (A - b_{11}\, I)x_1 &= c_1, \\ (A - b_{22}\, I)x_2 &= x_1\, b_{12} + c_2, \\ (A - b_{33}\, I)x_3 &= x_1\, b_{13} + b_{23}\, x_2 + c_3, \\ &\vdots \end{aligned}$$

The diagonal elements b_{kk} are the eigenvalues of B which are, by assumption, different from the eigenvalues of A. Thus, the matrices $A - b_{kk}\, I$ are nonsingular and the linear system is uniquely solvable.

In the general case use the fact that for any nonsingular T of size $m \times m$,

$$AX - XB = C \quad \Leftrightarrow \quad A(XT) - (XT)(T^{-1}BT) = CT.$$

Choose T with entries in an algebraicly closed extension of $\mathbb{K}$ such that $T^{-1}BT$ is upper triangular and solve for XT. This is the basis for the **Bathels-Stewart algorithm**.

Main theorem on Sylvester operators: integral formula

Suppose $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$ have no common eigenvalue. Let Γ be a closed contour that surronds the eigenvalues of A counterclockwise and excludes the eigenvalues of B. (Γ may be a union of circles).

Then the unique solution X of

$$AX - XB = C$$

is given by the integral

$$X = \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1} C (\lambda I - B)^{-1}\, d\lambda. \qquad (*)$$

<u>Proof.</u> For X defined by $(*)$,

$2\pi i(AX - XB)$

$$= \int_\Gamma (A(\lambda I - A)^{-1} C (\lambda I - B)^{-1} - (\lambda I - A)^{-1} C (\lambda I - B)^{-1} B)\, d\lambda$$

$$= \int_\Gamma ((A - \lambda I)(\lambda I - A)^{-1} C (\lambda I - B)^{-1} + (\lambda I - A)^{-1} C (\lambda I - B)^{-1}(\lambda I - B))\, d\lambda$$

$$= \int_\Gamma ((\lambda I - A)^{-1} C - C(\lambda I - B)^{-1})\, d\lambda = -C \underbrace{\int_\Gamma (\lambda I - B)^{-1}\, d\lambda}_{=0} + \underbrace{\int_\Gamma (\lambda I - A)^{-1} d\lambda}_{=2\pi i\, I}.$$

The identity $\int_\Gamma (\lambda I - B)^{-1}\, d\lambda$ holds since $(\lambda I - B)^{-1}$ holomorphic in the domain surrounded be Γ. The identity $\int_\Gamma (\lambda I - A)^{-1} d\lambda = 2\pi i\, I$ follows from the spectral theorem combined with the residue theorem: $A = \sum_k \lambda_k P_k + N_k \;\Rightarrow$

$(\lambda I - A)^{-1} = \sum_k (\lambda - \lambda_k)^{-1} P_k + \ldots \;\Rightarrow\; \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1} d\lambda = \sum_k P_k = I.$

Eigenvalues of Sylvester operators

The eigenvalues of $\mathcal{S}(X) = AX - XB$ are the differences of the eigenvalues of A and B.

More precisely:

Let $\lambda_1^A, \ldots, \lambda_n^A$ be the eigenvalues of A, let $\lambda_1^B, \ldots, \lambda_m^B$ be the eigenvalues of B.
Then the eigenvalues of $\mathcal{S}$ are $\lambda_j^A - \lambda_k^B$, $j = 1, \ldots, n$, $k = 1, \ldots, m$.

<u>Proof.</u>
Suppose $AX - XB = \lambda X$. Then $(A - \lambda I)X - XB = 0$. Thus, $(A - \lambda I)$ and B have a common eigenvalue. Hence $\lambda_j^A - \lambda = \lambda_k^B$ for some j, k.

On the other hand suppose

$$Av = \lambda_j^A v, \qquad w^\top B = \lambda_k^B w^\top.$$

Then

$$A(vw^\top) - (vw^\top)B = (\lambda_j^A - \lambda_k^B)(vw^\top).$$

Thus, $vw^\top$ is an eigenvector of $\mathcal{S}$ to the eigenvalue $\lambda_j^A - \lambda_k^B$.

<u>Problem (the author of these notes does not know the answer):</u>
What is the Jordan canonical form of $\mathcal{S}$?

An application: pole placement

Remainder.

Linear system: $\dot{x} = Ax + Bu$. Feedback: $u = -Fx + v$. closed loop: $\dot{x} = (A - BF)x + v$

Let A and L be square matrices of the same size. Suppose

$$AX - XL = BU$$

If X is nonsingular then multiplication with X^{-1} from the right yields

$$A - B\underbrace{UX^{-1}}_{=:F} = XLX^{-1} \qquad (F = \text{ feedback})$$

Thus the closed loop system $A - BF$ has the same eigenvalues as L.

<u>Obstacle:</u> nonsingularity of X.

More about pole placement in the next slide collection.

An application: block diagonalization of block triangular matrices

We have

$$\underbrace{\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}}_{\mathcal{X}^{-1}} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \underbrace{\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}}_{\mathcal{X}} = \underbrace{\begin{bmatrix} A & AX - XB + C \\ 0 & B \end{bmatrix}}_{\mathcal{M}}.$$

The upper right block of $\mathcal{M}$ vanishes iff

$$AX - XB = -C.$$

This Sylvester equation has a unique solution iff A and B have no common eigenvalue.

The Sylvester differential equation: block diagonalization of block triangular ODE

Consider the ODE: $\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

Substitution: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$. This implies

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 0 & \dot{X} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$\Leftrightarrow$

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}^{-1} \left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - \begin{bmatrix} 0 & \dot{X} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right) \\ &= \underbrace{\begin{bmatrix} A & AX - XB + C - \dot{X} \\ 0 & B \end{bmatrix}}_{\mathcal{M}} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} f_1 - X f_2 \\ f_2 \end{bmatrix}. \end{aligned}$$

The upper right block of $\mathcal{M}$ vanishes iff X solves the **Sylvester differential equation**

$$\dot{X} = AX - XB + C.$$

The Sylvester differential equation: Solution

Let $A(t) \in \mathbb{C}^{n\times n}$, $B(t) \in \mathbb{C}^{m\times m}$, $C(t) \in \mathbb{C}^{n\times m}$ be piecewise continuous matrix functions.

Let $\Phi_A(t) \in \mathbb{C}^{n\times n}$ $\Phi_B(t) \in \mathbb{C}^{m\times m}$ be such that

$$\dot{\Phi}_A(t) = A(t)\,\Phi_A(t), \qquad \dot{\Phi}_B(t) = \Phi_B(t)\,(-B(t)), \qquad \Phi_A(t_0) = I, \qquad \Phi_B(t_0) = I.$$

Then the unique solution of

$$\dot{X} = AX - XB + C, \qquad X(t_0) = X_0$$

is

$$X(t) = \Phi_A(t) X_0 \Phi_B(t) + \int_{t_0}^{t} \Phi_A(t,s) C(s) \Phi_B(t,s)\, ds,$$

where $\Phi_A(t,s) = \Phi_A(t)\Phi_A(s)^{-1}$, $\Phi_B(t,s) = \Phi_B(s)^{-1}\Phi_B(t)$.

If A and B are both constant then

$$X(t) = e^{A(t-t_0)} X_0\, e^{-B(t-t_0)} + \int_{t_0}^{t} e^{A((t-s)} C(s) e^{-B(t-s)}\, ds.$$

<u>Proof.</u> direct computation.

Lyapunov operators

Lyapunov operators and Lyapunov equations

Lyapunov operator:

$$\mathcal{L}_A : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}, \qquad \mathcal{L}_A(P) := A^*P + PA, \qquad \text{where } A \in \mathbb{C}^{n\times n}.$$

Lyapunov equations:

<u>algebraic:</u> $A^*P + PA = \pm Q,$ <u>differential:</u> $\dot{P} = A^*P + PA + Q.$

Lyapunov operators and the associated equation are special cases of Sylvester operators and the associated equations. First consequence:

The algebraic Lyapunov equation has a unique solution iff $\lambda + \bar{\nu} \neq 0$ for any pair of eigenvalues λ, ν of A. If this is the case and Q is Hermitian then P is Hermitian.

If the solution $P(\cdot)$ to a Lyapunov ODE with (pointwise) Hermitian Q is Hermitian at one time t_0 then $P(t)$ is Hermitian for all t. Proof: If $P(\cdot)$ is solution of the ODE then $P(\cdot)^*$ is also a solution. By uniqueness: $P(\cdot) = P(\cdot)^*$ if $P(t_0) = P(t_0)^*$.

Aleksandr Mikhailovich Lyapunov (1857-1918). Russian Mathematician.

Linear ODE, sesquilinear forms and Lyapunov operators

For a fixed matrix $P \in \mathbb{C}^{n\times n}$ we consider the **sesquilinear form**

$$\mathbb{C}^n \times \mathbb{C}^n \ni (v,w) \longmapsto v^* P w \in \mathbb{C}.$$

A nonsingular matrix $M \in \mathbb{C}^{n\times n}$ is called an **isometry** with respect to P if

$$(Mv)^* P (Mw) = v^* P w \qquad \text{for all } v, w.$$

The latter it equivalent to $M^* P M = P$.

Lemma. Let x, y be such that $\dot{x} = A\,x$, $\dot{y} = A\,y$. (A may be time depentent). Then

$$\frac{d}{dt}(x^* P y) = \dot{x}^* P y + x^* P \dot{y} = (Ax)^* P y + x^* P A y = x^*(A^* P + PA) y.$$

Corollary 1. If $A^* P + PA = 0$ then $x^* P y$ is constant (independent of t).

Corollary 2. Let $t \mapsto A(t)$ be continuous. The following are equivalent.

(1) $A(t)^* P + PA(t) = 0$ for all $t \in [a,b]$.

(2) The transition matrices $\Phi(t,t_0)$ associated with A are isometries:

$$(\Phi(t,t_0)v)^* P (\Phi(t,t_0)w) = v^* P w, \qquad \text{for all } t \in [a,b],\ v,w \in \mathbb{C}^n.$$

<u>Proof.</u> homework.

Corollary 3. Let $A \in \mathbb{C}^{n\times n}$. The following are equivalent.

(1) $A^*P + PA = 0$.

(2) The matrices e^{At} are isometries with respect to P:

$$(e^{At}v)^*P(e^{At}w) = v^*Pw, \qquad \text{for all } t \in [a,b],\ \ v,w \in \mathbb{C}^n.$$

Special case: The isometries with respect to $P = I$ are called **unitary**.
Thus, $U \in \mathbb{C}^{n\times n}$ is unitary iff $U^*U = I$.

Corollary 4. e^{At} is unitary iff A is **skew Hermitian**, i.e. $A^* = -A$, equivalently, $A = i\,H$ for some Hermitian H ($H = H^*$).

Special case for even n: The isometries with respect to $P = \mathcal{J} := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ are called **symplectic**. Thus, $S \in \mathbb{C}^{n\times n}$ is symplectic iff $S^*\mathcal{J}S = \mathcal{J}$. Note: $\mathcal{J}^* = -\mathcal{J}$.

Corollary 5. e^{At} is symplectic iff A is **Hamiltonian**, i.e. $A^*\mathcal{J} = -\mathcal{J}A$, equivalently, $\mathcal{J}A$ is Hermitian. $(\mathcal{J}A)^* = \mathcal{J}A$.

Main theorem on Lyapunov equations and stability

Part 1. Suppose $A \in \mathbb{C}^{n\times n}$ is Hurwitz (stable, i.e. all of its eigenvalues have negative real part). Then for all (not necessarily Hermitian) $Q \in \mathbb{C}^{n\times n}$ the unique solution of $A^*P + PA = -Q$ is

$$P = \int_0^\infty e^{A^*t} Q e^{At}\, dt. \qquad (*)$$

Proof. Since A is Hurwitz $\|e^{At}\| \le c\, e^{-\alpha\, t}$ for some $\alpha < 0$. Hence, the integral in $(*)$ exists. We have

$$\begin{aligned} A^*P + PA &= \int_0^t A^* e^{A^*t} Q e^{At} + A^* e^{A^*t} Q e^{At} A\, dt \\ &= \int_0^t \frac{d}{dt} e^{A^*t} Q e^{At}\, dt = \underbrace{\lim_{t\to\infty} (e^{A^*t} Q e^{At})}_{=0} - Q. \end{aligned}$$

Part 2. The following are equivalent

(a) $A \in \mathbb{C}^{n\times n}$ is Hurwitz.

(b) $\exists Q > 0 :\ A^*P + PA = -Q$ has a solution $P > 0$.

(c) $\forall Q > 0 :\ A^*P + PA = -Q$ has a solution $P > 0$.

Proof. $(a) \Rightarrow (c)$ follows from Part 1 since $e^{A^*t} = (e^{At})^*$. $(c) \Rightarrow (b)$ is trivial. $(b) \Rightarrow (a)$: Let $Av = \lambda v$, $V \neq 0$, and $A^*P + PA = -Q$. Then

$$-v^*Qv = v^*(A^*P + PA)v = (\bar{\lambda} + \lambda)\, v^*Pv = 2\,\Re(\lambda)\, v^*Pv \quad \Rightarrow \quad \Re(\lambda) < 0.$$

Digression: Diagonalization of Hermitian matrix pair (one matrix positive definite)

Theorem. Let $P, Q \in \mathbb{K}^{n\times n}$ be Hermitian, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $P > 0$.
Then there exists a basis $V = [v_1, \ldots, v_n]$ of $\mathbb{K}^n$ and real numbers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ such that

$$Qv_k = \lambda_k Pv_k, \quad (*) \qquad v_j^* Pv_k = \delta_{jk}, \quad (**) \qquad j, k = 1, \ldots, n.$$

Furthermore, for any $v \in \mathbb{K}^n \setminus \{0\}$,

$$\lambda_n \leq \frac{v^*Qv}{v^*Pv} \leq \lambda_1. \qquad \text{(Rayleigh quotent)}$$

Bounds are attained for $v = v_n$ and $v = v_1$.

Remarks: $(*) \Rightarrow$ the λ_k are the eigenvalues of $P^{-1}Q$. $(**)$ states that the eigenvectors can be chosen such that they form an orthonormal basis with respect to P. Definiteness of P is essential for the theorem. $P = I$ is standard result.

Proof sketch. Suppose $Qv = \lambda Pv$, $v \neq 0$. Then

$$\underbrace{v^*Qv}_{\in\mathbb{R}} = \lambda \underbrace{v^*Pv}_{>0} \qquad \Rightarrow \qquad \lambda \in \mathbb{R}.$$

Eigenvectors to different eigenvalues are P-orthogonal:

$$Qv = \lambda Pv,\ Qw = \nu Pw,\ \lambda \neq \nu \qquad \Rightarrow \qquad \lambda v^*Pw = v^*Qw = v^*Pw\,\mu \quad \Rightarrow \quad v^*Pw = 0.$$

Statement on Rayleigh quotient follows since each v is linear combination of the v_k.

Diagonalization of Hermitian matrix pair: Application to ODE of second order.

Proposition: With the assumption and notation of the theorem on the foregoing slide, and additionally $Q > 0$: The initial value problem

$$P\ddot{x} + Qx = 0, \qquad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0$$

has the unique solution

$$x(t) = \sum_{k=1}^{n} \left(\alpha_k \cos\left(\sqrt{\lambda_k}\, t\right) + \beta_k \sin\left(\sqrt{\lambda_k}\, t\right)\right) v_k,$$

where

$$\alpha_k = v_k^* P x_0, \qquad \beta_k = v_k^* P \dot{x}_0 / \sqrt{\lambda_k}.$$

Proof: Homework.

Diagonalization of Hermitian matrix pair: Growth bound for solutions of linear ODE

Theorem. Let $\beta \in \mathbb{R}$ be such that $A - \beta I$ is Hurwitz (stable). Let $Q > 0$, and let $P > 0$ be the solution of

$$(A - \beta I)^* P + P(A - \beta I) = -Q.$$

Let

$$\lambda_{\min} := \min_{v \neq 0} \frac{v^* Q v}{v^* P v}, \qquad \|v\|_P := \sqrt{v^* P v}.$$

Then the solution of

$$\dot{x} = Ax \qquad x(0) = x_0$$

satisfies

$$\|x(t)\|_P = e^{(\beta - \lambda_{\min}/2)\, t} \, \|x_0\|_P.$$

<u>Proof.</u> We have

$$\begin{aligned} \frac{d}{dt} x^* P x &= x^*(A^* P + PA)x \\ &= 2\beta\, x^* P x + x^*[(A - \beta I)^* P + P(A - \beta I)]x \\ &= 2\beta x^* P x - x^* Q x \leq (2\beta - \lambda_{\min})\, x^* P x. \end{aligned}$$

$\Rightarrow$ (Gronwall-Lemma)

$$x^* P x \leq e^{(2\beta - \lambda_{\min})\, t} \, x_0^* P x_0.$$

Basics on Riccati equations

Graph subspaces

The graph subspace $\mathcal{G}$ associated with $S \in \mathbb{K}^{m\times n}$ is defined as

$$\mathcal{G} = \left\{ \begin{bmatrix} x \\ Sx \end{bmatrix} \;\middle|\; x \in \mathbb{K}^n \right\} \subset \mathbb{K}^{n+m}$$

Since $\begin{bmatrix} x \\ Sx \end{bmatrix} = \begin{bmatrix} I \\ S \end{bmatrix} x$ the columns of $\begin{bmatrix} I \\ S \end{bmatrix}$ form a basis of $\mathcal{G}$.

Any other basis of $\mathcal{G}$ is of the form $\begin{bmatrix} X \\ SX \end{bmatrix} = \begin{bmatrix} I \\ S \end{bmatrix} X$ with nonsingular $X \in \mathbb{K}^{n\times n}$

<u>On the other hand:</u>

If X is nonsingular then the columns of $\begin{bmatrix} X \\ Y \end{bmatrix}$ form a basis of a graph subspace,

since $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} I \\ YX^{-1} \end{bmatrix} X$. Define $S = YX^{-1}$.

Graph subspaces as invariant subspaces and Riccati equations

The graph subspace $\begin{bmatrix} I \\ S \end{bmatrix}$ is an invariant subspace of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ iff

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I \\ S \end{bmatrix} = \begin{bmatrix} I \\ S \end{bmatrix} L \qquad \text{for some } L. \qquad (*)$$

The latter equation can be split into

$$A + BS = L, \qquad C + DS = SL. \qquad (**)$$

Replacing L in the second equation with the left hand side of the first yields the

Algebraic Riccati equation: $\qquad C + DS - SA - SBS = 0.$

On the other hand:
If S solves the Riccati equation and we define $L := A + BS$, then $(*)$ and $(**)$ hold.

Thus we have bijection: Invariant graph subspaces $\leftrightarrow$ solutions of Riccati equation.

Note: If we regard S as feedback then L is the matrix of the closed loop system. However, in practice (as we will see) instead of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ the more complicated matrices $\begin{bmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{bmatrix}$ are used to construct the feedback $F = -R^{-1}B^*S$.

An existence result

For small B the **Riccati equation:**

$$C + DS - SA - SBS = 0 \qquad (*)$$

can be regarded as a perturbed Sylvester equation $C + DS - SA = 0$.
This leads to the following result.

Theorem (Stewart, 1973).

Let $\|\cdot\|$ be any submultiplicative norm ($\|XY\| \leq \|X\|\,\|Y\|$).
Define the separation of A and D by

$$\text{sep}(A, D) := \inf\{\, \|DS - SA\| \;\mid\; \|S\| = 1\}.$$

If A and D have no common eigenvalue and $\|C\|\,\|B\| \leq \text{sep}(A, D)^2/4$
then $(*)$ has solution S with

$$\|S\| \leq \frac{2\,\|C\|}{\text{sep}(A, D)}. \qquad (**)$$

There is only one solution satisfying this inequality.

<u>Proof sketch.</u> Write $(*)$ as fixed point problem:

$$S = \underbrace{\text{Sylv}^{-1}(SBS - C)}_{=:T(S)}, \qquad \text{where} \qquad \text{Sylv}(S) := DS - SA.$$

The operator norm of Sylv^{-1} is $\|\text{Sylv}^{-1}\| = 1/\text{sep}(A, D)$. Verify that the operator T maps the set $\mathcal{B}$ of matrices S, which fulfill $(**)$, into itself. Show that T is a contraction on $\mathcal{B}$. Apply the contraction mapping theorem.

Application of Stewart's result:
Perturbation bound for invariant subspaces

Suppose a matrix $\mathcal{A} \in \mathbb{C}^{(n+m)\times(n+m)}$ has an invariant subspace $\mathcal{U}$ of dimension n. After a similarity transformation (suitable choice of basis) we may assume that

$$\mathcal{A} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \qquad \mathcal{U} = \mathrm{Im} \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{C}^{(n+m)\times n}.$$

Perturbation:

$$\mathcal{A} \rightsquigarrow \mathcal{A} + E = \begin{bmatrix} A + E_{11} & B + E_{12} \\ E_{21} & D + E_{22} \end{bmatrix}$$

Stewart:

If $\|B + E_{12}\| \, \|E_{21}\| \leq \mathrm{sep}(A + E_{11}, D + E_{22})^2/4$ then $\mathcal{A} + E$ has invariant subspace

$$\mathrm{Im} \begin{bmatrix} I \\ S \end{bmatrix} \in \mathbb{C}^{(n+m)\times n} \qquad \text{with} \qquad \|S\| \leq \frac{2\,\|E_{21}\|}{\mathrm{sep}(A + E_{11}, D + E_{22})}.$$

Application: block triagonalization of block matrices

We have

$$\underbrace{\begin{bmatrix} I & 0 \\ -S & I \end{bmatrix}}_{\mathcal{S}^{-1}} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \underbrace{\begin{bmatrix} I & 0 \\ S & I \end{bmatrix}}_{\mathcal{S}} = \underbrace{\begin{bmatrix} A + BS & B \\ \mathsf{Ric}(S) & D - SB \end{bmatrix}}_{\mathcal{M}},$$

where

$$\mathsf{Ric}(S) := C + DS - SA - SBS.$$

The lower left block of $\mathcal{M}$ vanishes iff

$$\mathsf{Ric}(S) = 0.$$

The Riccati differential equation: block triangularization of linear ODE

Consider the ODE: $\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} + \begin{bmatrix}f_1\\ f_2\end{bmatrix}$. $\quad$ (A, B, C, D, f_k may depend on time)

Substitution: $\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}I & 0\\ S & I\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}$. This implies.

$$\begin{bmatrix}I & 0\\ S & I\end{bmatrix}\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} + \begin{bmatrix}0 & 0\\ \dot{S} & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}I & 0\\ S & I\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}f_1\\ f_2\end{bmatrix}$$

$\Leftrightarrow$

$$\begin{aligned}\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} &= \begin{bmatrix}I & 0\\ S & I\end{bmatrix}^{-1}\left(\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}I & 0\\ S & I\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} - \begin{bmatrix}0 & 0\\ \dot{S} & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}f_1\\ f_2\end{bmatrix}\right)\\ &= \underbrace{\begin{bmatrix}A+BS & B\\ \mathrm{Ric}(S)-\dot{S} & D-SB\end{bmatrix}}_{\mathcal{M}}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}f_1\\ f_2 - Sf_1\end{bmatrix}.\end{aligned}$$

The lower left block of $\mathcal{M}$ vanishes iff S satisfies the **Riccati differential equation**

$$\dot{S} = \mathrm{Ric}(S) = C + DS - SA - SBS. \qquad (*)$$

General Problem: Solutions of $(*)$ may tend to ∞ in finite time (finite escape time). Thus, solutions may exist only on a subinterval of the domain of the linear ODE.

Digression: Fundamental solution of block triangular ODE

On the slide before we showed how to transform a linear ODE to block triangular form. Here is a formula for the fundamental solution of such an ODE.

Proposition. The ODE

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

has the transition matrices

$$\underbrace{\begin{bmatrix} \Phi_1(t,t_0) & \Phi_{12}(t,t_0) \\ 0 & \Phi_2(t,t_0) \end{bmatrix}}_{\Phi},$$

(i.e. $\dot{\Phi} = \mathcal{A}\,\Phi,\ \ \Phi(t_0,t_0) = I$) where

$$\dot{\Phi}_k(t,t_0) = A_k(t)\,\Phi_k(t,t_0), \quad \Phi_k(t_0,t_0) = I, \qquad \text{(transition matrix of } A_k) \quad k = 1,2$$

and

$$\Phi_{12}(t,t_0) = \int_{t_0}^{t} \Phi_1(t,s)\,A_{12}(s)\,\Phi_2(s,t_0)\,ds.$$

The matrix function Φ_{12} satisfies the inhomogeneous ODE

$$\dot{\Phi}_{12} = A_1\,\Phi_{12} + A_{12}\Phi_2, \qquad \Phi_{12}(t_0,t_0) = 0.$$

If $t_0 = 0$ and $\mathcal{A}$ is constant then $\Phi_k(t) = e^{A_k t}$, $\quad \Phi_{12}(t) = \int_0^t e^{A_1(t-s)}\,A_{12}\,e^{A_2 s}\,ds.$

<u>Proof.</u> homework.

Hamiltonian matrices and Riccati equations in the narrow sense

In control theory the most important Riccati equations are

$$0 = Q + A^*S + SA - S(BR^{-1}B^*)S, \qquad R = R^* > 0,\ Q = Q^* \geq 0.$$

since these equations appear in optimal control problem, see the next slide collections. The interest is in positive Hermitian solutions S. These are Riccati equations in the narrow sense. They are the equations associated with the Hamiltonian matrices

$$H = \begin{bmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{bmatrix}.$$

Hamiltonian matrices and Riccati equations in the narrow sense

Consider the Hamiltonian matrix

$$H = \begin{bmatrix} A & -\widehat{R} \\ -Q & -A^* \end{bmatrix}, \qquad \widehat{R} \text{ and } Q \text{ Hermitian.}$$

It satisfies $H^*J + JH = 0$, where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. This yields the following.

Spectral decomposition of Hamiltonian matrices:

If $\lambda \in \mathbb{C}$ is an eigenvalue of H then $-\bar{\lambda}$ is also an eigenvalue of H.

Moreover, H has the spectral decomposition

$$H = \sum_{k=1}^{m} (\lambda_k\, P_k + N_k) + \sum_{k=1}^{m} (-\bar{\lambda}_k\; J^{-1} P_k^* J - J^{-1} N_k^* J) + \sum_{k=1}^{\ell} (i\,\omega_k\, P_{m+k} + N_{m+k}) \qquad (*)$$

where $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ are the distinct eigenvalues of H with negative real part, and $i\,\omega_1, \ldots, i\,\omega_\ell \in i\,\mathbb{R}$ are the imaginary eigenvalues (if there are any).

Since the spectral projectors P_k and $J^{-1} P_k^* J$ have the same rank the H-invariant subspaces

$$\mathcal{H}_- = \bigoplus_{k=1}^{m} \mathrm{Ker}\,(\lambda_k\, I - H)^{m_k}, \qquad \mathcal{H}_+ = \bigoplus_{k=1}^{m} \mathrm{Ker}\,(-\bar{\lambda}_k\, I - H)^{m_k},$$

associated with the stable and unstable eigenvalues have the same dimension.

If H has no imaginary eigenvalues and $H \in \mathbb{C}^{2n \times 2n}$ then the dimension of $\mathcal{H}_-$ is n.

Remark: $(*)$ follows from the relation $H = J^{-1}(-H^*)J$.

Hamiltonian matrices and Riccati equations in the narrow sense

Hamiltonian matrices and the associated Riccati equations can be used to construct various stabilizing feedbacks:

Let

$$H := \begin{bmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{bmatrix} \in \mathbb{C}^{2n\times 2n}, \qquad R = R^* > 0,\ Q = Q^* \geq 0.$$

Suppose H has no eigenvalues on the imaginary axis. Then the invariant subspace

$$\mathcal{H}_- = \bigoplus_{k=1}^{m} \mathrm{Ker}\,(\lambda_k I - H)^{m_k}$$

associated with the stable eigenvalues has dimension n. Suppose further, that $\mathcal{H}_-$ is a graph subspace with basis $\begin{bmatrix} X \\ Y \end{bmatrix}$. Then $S := YX^{-1}$ is the unique Hermitian and positive definite solution of the Riccati equation

$$0 = Q + A^*S + SA - S(BR^{-1}B^*)S,$$

and

$$\begin{bmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} I \\ S \end{bmatrix} = \begin{bmatrix} I \\ S \end{bmatrix} (A - BR^{-1}B^*S), \qquad A - BR^{-1}B^*S \text{ is Hurwitz.}$$

Thus, $F := -R^{-1}B^*S$ is a stabilizing feedback of $\dot{x} = Ax + Bu$.

Remark: this feedback solves an optimal control problem (for infinite time horizon). See the next slide collection.

A connection between Sylvester and Riccati equations

Suppose that a solution $t \to X(t) \in \mathbb{C}^{n \times n}$ to the Sylvester ODE

$$\dot{X} = AX - XB + C \qquad (*)$$

is nonsingular. Then its inverse $S(t) := X(t)^{-1}$ satisfies the Riccati ODE

$$-\dot{S} = SA - BS + SCS. \qquad (**)$$

<u>Proof.</u> Multiplying $(*)$ with S from the left and from the right yields

$$S\dot{X}S = SA - BS + SCS.$$

Differentiating $SX = I$ we infer

$$0 = \dot{S}\,X + S\,\dot{X} \quad \Rightarrow \quad S\dot{X}S = -\dot{S}.$$

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