GIAN course: Singular optimal control

Slide collection 4

Matrix Polynomials and ODE/DAE of higher order

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Subject of these notes

Matrix polynomial of degree d (with quadratic coefficients):

$$P(\lambda) = P_d \lambda^d + P_{d-1} \lambda^{d-1} + \ldots + P_1 \lambda + P_0, \qquad P_k \in \mathbb{C}^{n \times n}, \quad P_d \neq 0.$$

We investigate the solutions of the associated differential equation/DAE

$$P_d x^{(d)}(t) + P_{d-1} x^{(d-1)}(t) + \ldots + P_1 \dot{x}(t) + P_0 x(t) = f(t).$$

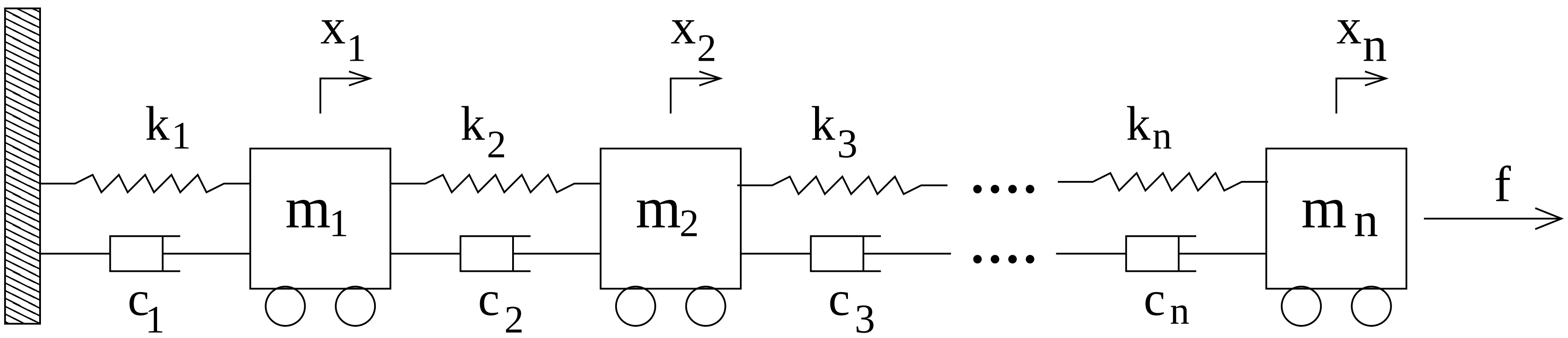
In short notation:

$$P\left(\frac{d}{dt}\right) x(t) = f(t),$$

where

$$P\left(\frac{d}{dt}\right) = P_d \left(\frac{d}{dt}\right)^d + P_{d-1} \left(\frac{d}{dt}\right)^{d-1} + \ldots + P_1 \left(\frac{d}{dt}\right) + P_0.$$

Example: mass-spring-damper chain



Equation of motion:

$$\begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{bmatrix} + \begin{bmatrix} c_1 & -c_2 & & & \\ -c_2 & c_2 + c_3 & -c_3 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -c_{n-1} \\ & & & -c_{n-1} & c_n \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}$$

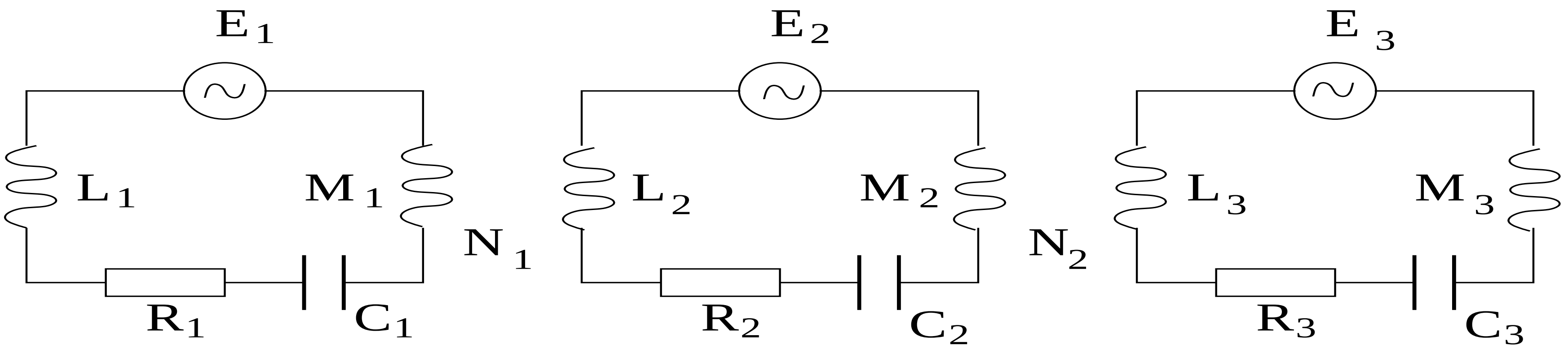
$$+ \begin{bmatrix} k_1 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -k_{n-1} \\ & & & -k_{n-1} & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f \end{bmatrix}$$

Short:

$$M\,\ddot{x} + C\,\dot{x} + K\,x = \hat{f}.$$

M=mass matrix, C=damping matrix, K=stiffness matrix, $\hat{f}$=external force.

Example: inductively coupled circuits (found in book by Veselic)



ODE for currents:

$$\begin{bmatrix} L_1+M_1 & N_1 & & & \\ N_1 & L_2+M_2 & N_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & N_{n-1} \\ & & & N_{n-1} & L_n+M_n \end{bmatrix} \begin{bmatrix} \ddot{I}_1 \\ \ddot{I}_2 \\ \vdots \\ \ddot{I}_n \end{bmatrix} + \begin{bmatrix} R_1 & & & & \\ & R_2 & & & \\ & & \ddots & & \\ & & & R_{n-1} & \\ & & & & R_n \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \vdots \\ \dot{I}_n \end{bmatrix}$$

$$+ \begin{bmatrix} C_1^{-1} & & & & \\ & C_2^{-1} & & & \\ & & \ddots & & \\ & & & C_{n-1}^{-1} & \\ & & & & C_n^{-1} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} \dot{E}_1 \\ \vdots \\ \dot{E}_{n-1} \\ \dot{E}_n \end{bmatrix}$$

Short: $\mathcal{L}\,\ddot{I} + \mathcal{R}\,\dot{I} + \mathcal{K}\,I = \dot{E}.$

L_k, M_k inductance, N_k mutual inductance, R_k resistance, C_k capacity.

The first companion form

The ODE/DAE

$$P_d\, x^{(d)} + P_{d-1}\, x^{(d-1)} + \ldots + P_1\, \dot{x} + P_0\, x = f \qquad (*)$$

is equivalent to the linear ODE/DAE

$$\begin{bmatrix} I & & & & \\ & I & & & \\ & & \ddots & & \\ & & & I & \\ & & & & P_d \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(d-1)} \end{bmatrix} - \underbrace{\begin{bmatrix} 0 & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \\ -P_0 & -P_1 & \ldots & \ldots & -P_{d-1} \end{bmatrix}}_{=:C_P^\flat} \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(d-1)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f \end{bmatrix} \qquad (**)$$

This is an ODE/DAE associated with the **companion pencil**

$$\underbrace{\lambda \begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & P_d \end{bmatrix} - \begin{bmatrix} 0 & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \\ -P_0 & -P_1 & \ldots & \ldots & -P_{d-1} \end{bmatrix}}_{C_P^\flat(\lambda)}$$

Precisely: if $y(t) = \begin{bmatrix} y_0(t) \\ y_1(t) \\ \vdots \\ y_{d-1}(t) \end{bmatrix}$ solves $C_P^\flat(\frac{d}{dt})y = [0 \ldots 0\, f^\top]^\top$ then $x(t) := y_0(t)$ solves $(*)$.

The first companion form

The ODE/DAE

$$P_d\, x^{(d)} + P_{d-1}\, x^{(d-1)} + \ldots + P_1\, \dot{x} + P_0\, x = f \qquad (*)$$

is equivalent to the linear ODE/DAE

$$\begin{bmatrix} I & & & & \\ & I & & & \\ & & \ddots & & \\ & & & I & \\ & & & & P_d \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(d-1)} \end{bmatrix} - \underbrace{\begin{bmatrix} 0 & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \\ -P_0 & -P_1 & \ldots & \ldots & -P_{d-1} \end{bmatrix}}_{=: C^\flat(P)} \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(d-1)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f \end{bmatrix} \qquad (**)$$

If P_d is nonsingular,

$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(d-1)} \end{bmatrix} - \begin{bmatrix} 0 & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \\ -\widehat{P}_0 & -\widehat{P}_1 & \ldots & \ldots & -\widehat{P}_{d-1} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(d-1)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ \widehat{f} \end{bmatrix}, \qquad \text{where} \quad \begin{cases} \widehat{P}_k := P_d^{-1} P_k, \\ \widehat{f} := P_d^{-1} f. \end{cases}$$

Thus, x solves $(*)$ if and only if

$$x(t) = \begin{bmatrix} I & 0 & \ldots & 0 \end{bmatrix} \left(e^{C^\flat(\widehat{P})\,(t-t_0)} \begin{bmatrix} x(t_0) \\ \dot{x}(t_0) \\ \vdots \\ x^{(d-1)}(t_0) \end{bmatrix} + \int_{t_0}^{t} e^{C^\flat(\widehat{P})\,(t-s)} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \widehat{f}(s) \end{bmatrix} ds \right)$$

Remark: in $(**)$ the identitiy matrix can be replaced by any nonsingular matrix.

Horner polynomials (my terminology) and division by a linear scalar factor

The Horner polynomials associated with $P(\lambda) = \sum_{k=0}^{d} P_k \lambda^k$ are defined as

$$\begin{aligned}
H_{d-1}(\lambda) &:= P_d, \\
H_{d-2}(\lambda) &:= P_d \lambda + P_{d-1}, \\
H_{d-3}(\lambda) &:= P_d \lambda^2 + P_{d-1} \lambda + P_{d-2}, \\
&\vdots \\
H_1(\lambda) &:= P_d \lambda^{d-2} + P_{d-1} \lambda^{d-3} + P_{d-2} \lambda^{d-4} + \ldots + P_2, \\
H_0(\lambda) &:= P_d \lambda^{d-1} + P_{d-1} \lambda^{d-2} + P_{d-2} \lambda^{d-3} + \ldots + P_2 \lambda + P_1.
\end{aligned}$$

Then it holds that (evaluation of $P(\lambda)$ by Horner's method)

$$\begin{aligned}
H_{d-1}(\lambda) &= P_d, \\
H_{k-1}(\lambda) &= H_k(\lambda) \lambda + P_k, \text{ for } k = d-1, \ldots, 1, \\
P(\lambda) &= H_0(\lambda) \lambda + P_0.
\end{aligned}$$

Furthermore, it is easily verified that for any $\lambda_0 \in \mathbb{C}$,

$$P(\lambda) = \left(\sum_{k=0}^{d-1} H_k(\lambda_0) \lambda^k \right) (\lambda - \lambda_0) + P(\lambda_0).$$

Digression: Horner polynomials and $(\lambda I - A)^{-1}$

Let χ_A be the characteristic polynomial of $A \in \mathbb{C}^{n\times n}$. Then

$$\chi_A(\lambda) = \left(\sum_{k=0}^{n-1} H_k(\lambda_0)\,\lambda^k\right)(\lambda - \lambda_0) + \chi_A(\lambda_0),$$

where the H_k are the associated Horner polynomials. By Cayley Hamilton:

$$0 = \chi_A(A) = \left(\sum_{k=0}^{n-1} H_k(\lambda_0)\,A^k\right)(A - \lambda_0 I) + \chi_A(\lambda_0)\,I.$$

Thus,

$$(\lambda_0 I - A)^{-1} = \frac{1}{\chi_A(\lambda_0)}\left(\sum_{k=0}^{n-1} H_k(\lambda_0)\,A^k\right).$$

The characteristic polynomial

The polynomial $\chi_P(\lambda) = \det(P(\lambda))$ is called the characteristic polynomial of $P(\lambda) = \sum_{k=0}^{d} P_k\,\lambda^k \in \mathbb{C}^{n\times n}[\lambda]$.

Note that degree$(\chi_P) \le d\,n$ with equality if P_d is nonsingular.

Lemma. χ_P equals the char. polynomial of the companion pencil of P up to sign.

Proof. for $d=4$. The Horner polynomials $H_k(\lambda)$ satisfy

$$H_2(\lambda) = P_4\,\lambda + P_3, \quad H_1(\lambda) = H_2(\lambda)\,\lambda + P_2, \quad H_0(\lambda) = H_1(\lambda)\,\lambda + P_1, \quad P(\lambda) = H_0(\lambda)\,\lambda + P_0.$$

Thus,

$$\overbrace{\begin{bmatrix} H_0(\lambda) & H_1(\lambda) & H_2(\lambda) & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \end{bmatrix}}^{F(\lambda)} \left(\lambda \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & P_4 \end{bmatrix} - \begin{bmatrix} 0 & I & & \\ & & I & \\ & & & I \\ -P_0 & -P_1 & -P_2 & -P_3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} H_0(\lambda) & H_1(\lambda) & H_2(\lambda) & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \end{bmatrix} \begin{bmatrix} \lambda I & -I & & \\ & \lambda I & -I & \\ & & \lambda I & -I \\ P_0 & P_1 & P_2 & \lambda P_4 + P_3 \end{bmatrix} = \underbrace{\begin{bmatrix} P(\lambda) & & & \\ -\lambda I & I & & \\ & -\lambda I & I & \\ & & -\lambda I & I \end{bmatrix}}_{G(\lambda)}$$

We have $\det(F(\lambda)) = \pm 1$ (change first and fourth block row) and $\det(G(\lambda)) = P(\lambda)$.

The second companion form

For a **monic** polynomial $P(\lambda) = I\,\lambda^d + P_{d-1}\,\lambda^{d-1} + \ldots + P_1\,\lambda + P_0$ define an invertible coordinate transformation

$$(x, \dot{x}, \ddot{x}, \ldots, x^{(d-1)}) \longmapsto (z_0, z_1, \ldots, z_{d-1})$$

by

$$\begin{array}{rcllcl}
z_{d-1} & := & x & & = & H_{d-1}(\tfrac{d}{dt})x \\
z_{d-2} & := & \dot{x} + P_{d-1}\,x & & = & H_{d-2}(\tfrac{d}{dt})x \\
z_{d-3} & := & \ddot{x} + P_{d-1}\,\dot{x} + P_{d-2}x & & = & H_{d-3}(\tfrac{d}{dt})x \\
 & \vdots & & & \vdots & \\
z_0 & := & x^{(d-1)} + P_{d-1}x^{(d-2)} + \ldots + P_1\,x & & = & H_0(\tfrac{d}{dt})x
\end{array}$$

Then

$$\text{for } k = 1, \ldots, d-1: \quad \dot{z}_k = z_{k-1} - P_k\,z_{d-1}, \qquad \text{and} \qquad \dot{z}_0 = P(\frac{d}{dt})x - P_0\,z_{d-1}.$$

Thus,

$$P\left(\frac{d}{dt}\right)x = f \quad \Leftrightarrow \quad \frac{d}{dt}\begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{d-1} \end{bmatrix} - \underbrace{\begin{bmatrix} 0 & & & -P_0 \\ I & & & -P_1 \\ & \ddots & & \vdots \\ & & I & -P_{d-1} \end{bmatrix}}_{=:C_P^\sharp} \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{d-1} \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (*)$$

Thus, the solutions of $(*)$ satisfy

$$x(t) = \begin{bmatrix} 0 & \ldots & 0 & I \end{bmatrix} \left(e^{C_P^\sharp\,(t-t_0)} \begin{bmatrix} z_1(t_0) \\ z_2(t_0) \\ \vdots \\ z_d(t_0) \end{bmatrix} + \int_{t_0}^{t} e^{C_P^\sharp\,(t-s)} \begin{bmatrix} f(s) \\ 0 \\ \vdots \\ 0 \end{bmatrix} ds \right)$$

with the **second companion matrix** $C_P^\sharp$.

Similarity of companion matrices

It is straightforward to verify that

$$\underbrace{\begin{bmatrix} P_1 & P_2 & \dots & P_{n-1} & I \\ P_2 & & & I & \\ \vdots & & & & \\ P_{n-1} & I & & & \\ I & & & & \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} 0 & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \\ -P_0 & -P_1 & \dots & \dots & -P_{n-1} \end{bmatrix}}_{C_P^\flat} = \underbrace{\begin{bmatrix} 0 & & & -P_0 \\ I & & & -P_1 \\ & \ddots & & \vdots \\ & & I & -P_{n_1} \end{bmatrix}}_{C_P^\sharp} \underbrace{\begin{bmatrix} P_1 & P_2 & \dots & P_{n-1} & I \\ P_2 & & & I & \\ \vdots & & & & \\ P_{n-1} & I & & & \\ I & & & & \end{bmatrix}}_{T}$$

Thus,

$$C_P^\sharp = T\, C_P^\flat\, T^{-1}.$$

Note: T is an upper triangular Hankel matrix.

Jordan canonical form of companion matrices of scalar polynomials

The identity $p(\lambda) = \lambda^n + p_{n-1}\,\lambda^{n-1} + \ldots + p_1\,\lambda + p_0$ can be rewritten as

$$-p_{n-1}\,\lambda^{n-1} - \ldots - p_1\,\lambda - p_0 = \lambda^n - p(\lambda).$$

Thus,

$$\underbrace{\begin{bmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ -p_0 & -p_1 & \ldots & \ldots & -p_{n-1} \end{bmatrix}}_{C_p^\flat} \underbrace{\begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{bmatrix}}_{=:v_0(\lambda)} = \lambda \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ p(\lambda) \end{bmatrix} \qquad (*)$$

$v_0(\lambda)$ is an eigenvector of $C_p^\flat$ iff $p(\lambda) = 0$.
Suppose now, that λ is a zero of p of multiplicity $m > 1$. Then

$$0 = p(\lambda) = p'(\lambda) = \ldots = p^{(m-1)}(\lambda).$$

Differentiating $(*)$ yields

$$C_p^\flat \left(\frac{d}{d\lambda}\right)^j v_0(\lambda) = \left(\frac{d}{d\lambda}\right)^j (\,\lambda\, v_0(\lambda)\,), \qquad j = 0, \ldots, m-1.$$

Notation:

$$v_j(\lambda) := \frac{1}{j!}\left(\frac{d}{d\lambda}\right)^j v_0(\lambda) \;\;\Rightarrow\;\; \frac{1}{j!}\left(\frac{d}{d\lambda}\right)^j (\lambda\, v_0(\lambda)) = \lambda\, v_j(\lambda) + v_{j-1}(\lambda), \qquad j = 1, \ldots, m-1$$

$$\Rightarrow \;\; C_p^\flat\, v_j(\lambda) = \lambda\, v_j(\lambda) + v_{j-1}(\lambda)$$

$$\Rightarrow \;\; v_0(\lambda), \ldots, v_{m-1}(\lambda) \text{ is a Jordan chain for } C_p^\flat$$

Jordan canonical form of companion matrices of scalar polynomials

Example: Suppose

$$p(\lambda) = \lambda^4 + p_3\,\lambda^3 + p_2\,\lambda^2 + p_1\,\lambda + p_0 \;=\; (\lambda - \lambda_1)(\lambda - \lambda_2)^3, \qquad \lambda_1 \neq \lambda_2.$$

The Jordan chain to the zero λ_2 is

$$v_0(\lambda_2) = \begin{bmatrix} 1 \\ \lambda_2 \\ \lambda_2^2 \\ \lambda_2^3 \end{bmatrix}, \quad v_1(\lambda_2) = v_0'(\lambda_2) = \begin{bmatrix} 0 \\ 1 \\ 2\,\lambda_2 \\ 3\,\lambda_2^2 \end{bmatrix}, \quad v_2(\lambda_2) = \frac{1}{2}\,v_0''(\lambda_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3\lambda_2 \end{bmatrix}.$$

Let

$$V = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \lambda_1 & \lambda_2 & 1 & 0 \\ \lambda_1^2 & \lambda_2^2 & 2\lambda_2 & 1 \\ \lambda_1^3 & \lambda_2^3 & 3\lambda_2^2 & 3\lambda_2 \end{bmatrix}, \qquad J = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & 1 & \\ & & \lambda_2 & 1 \\ & & & \lambda_2 \end{bmatrix}.$$

Then

$$C_p^\flat = V\,J\,V^{-1}, \qquad C_P^\sharp = TV\;J\;(TV)^{-1} \qquad (\text{since } C_P^\sharp = T\,C_p^\flat\,T^{-1}).$$

V is a **confluent Vandemonde matrix**.

Remainder: Companion matrices and the Matrix exponential

Proposition. Suppose

$$\begin{bmatrix} \dot{\phi}_0 \\ \dot{\phi}_1 \\ \vdots \\ \dot{\phi}_{n-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & & & -p_0 \\ 1 & & & -p_1 \\ & \ddots & & \vdots \\ & & 1 & -p_{n_1} \end{bmatrix}}_{C_p^\sharp} \underbrace{\begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{n-1} \end{bmatrix}}_{\phi}, \qquad \begin{bmatrix} \phi_0(0) \\ \phi_1(0) \\ \vdots \\ \phi_{n-1}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then

$$p\left(\frac{d}{dt}\right)\phi_k = 0, \quad (*) \qquad \text{and} \qquad \phi_k^{(j)}(0) = \delta_{jk} \quad (**).$$

Thus, by our earlier results

$$p(A) = 0 \quad \Rightarrow \quad e^{At} = \sum_{k=0}^{n-1} \phi_k(t)\, A^k.$$

Note that the vector ϕ is the first column of $\exp(C_a^\sharp\, t)$.

Proof. $(*)$ follows from $p\left(\frac{d}{dt}\right) e^{C^\sharp t} = p(C^\sharp) e^{C^\sharp t} = 0$. $(**)$ is exercise.

Deflating pairs and standard pairs

Suppose the matrices L (quadratic) and V (may be rectangular) satisfy

$$0 = P_d V L^d + P_{d-1} V L^{d-1} + \ldots + P_1 V L + P_0 V.$$

Then (V, L) is called a **deflating pair** for P. We have

(1) for any vector ξ of compatible size: $\quad x(t) := V\, e^{L\,(t-t_0)}\, \xi \quad \Rightarrow \quad P(\frac{d}{dt})x = 0.$

(2)

$$\begin{bmatrix} 0 & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \\ -P_0 & -P_1 & \ldots & \ldots & -P_{d-1} \end{bmatrix} \underbrace{\begin{bmatrix} V \\ V\,L \\ \vdots \\ V\,L^{d-1} \end{bmatrix}}_{=:V_L} = \begin{bmatrix} V \\ V\,L \\ \vdots \\ P_d\,V\,L^{d-1} \end{bmatrix} L.$$

If V_L is nonsingular (which implies $V \in \mathbb{C}^{n\times nd}$, $L \in \mathbb{C}^{nd\times nd}$)
then (V, L) is called a **standard pair** for P.

Let (V, L) be a standard pair and suppose that P_d is nonsingular. Then $P(\frac{d}{dt})\,x = f$ if and only if

$$x(t) = V \left(e^{L\,(t-t_0)}\xi + \int_{t_0}^{t} e^{L\,(t-s)} B\, f(s)\, ds \right), \quad B := V_L^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ P_d^{-1} \end{bmatrix}, \xi = V_L^{-1} \begin{bmatrix} x(t_0) \\ \dot{x}(t_0) \\ \vdots \\ x^{(d-1)}(t_0) \end{bmatrix}.$$

Additivity of deflating pairs

If (V_1, L_1) and (V_2, L_2) are deflating pairs for P

then $([V_1, V_2], \mathrm{diag}(L_1, L_2)$ is a deflating pair for P.

Proof. The identities

$$
\begin{aligned}
0 &= P_d V_1 L_1^d + P_{d-1} V_1 L_1^{d-1} + \ldots + P_1 V_1 L_1 + P_0 V_1, \\
0 &= P_d V_2 L^d + P_{d-1} V_2 L_2^{d-1} + \ldots + P_1 V_2 L_2 + P_0 V_2
\end{aligned}
$$

are equivalent to

$$
0 = P_d [V_1, V_2] \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}^d + P_{d-1} [V_1, V_2] \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}^{d-1} + \ldots + P_1 [V_1, V_2] \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} + P_0 [V_1, V_2].
$$

□

Regularity, eigenvalues and eigenvectors

- The matrix Polynomial $P(\lambda) = \sum_{k=0}^{d} P_k \lambda^k \in \mathbb{C}^{n \times n}[\lambda]$ is said to be **regular**, if its **characteristic polynomial**

$$\chi(\lambda) := \det(P(\lambda))$$

  is not the zero polynomial. In this case its degree is $\leq d\, n$.
- (v, λ) is called an **eigenpair** to the polynomial P with **eigenvector** $v \in \mathbb{C}^n \setminus \{0\}$ and **eigenvalue** $\lambda \in \mathbb{C}$ if

$$P(\lambda)\, v = 0.$$

(1) The eigenvalues of P are the zeros of $\chi(\lambda)$.

(2) (v, λ) is an eigenpair if and only if

$$\begin{bmatrix} 0 & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \\ -P_0 & -P_1 & \dots & \dots & -P_{d-1} \end{bmatrix} \underbrace{\begin{bmatrix} v \\ v\,\lambda \\ \vdots \\ v\,\lambda^{d-1} \end{bmatrix}}_{=:q(v,\lambda)} = \begin{bmatrix} v \\ v\,\lambda \\ \vdots \\ P_d\, v\,\lambda^{d-1} \end{bmatrix} \lambda$$

(3) If $P(\lambda_i)\, v_i = 0$, $i = 1, \dots, m$ then

$$V = [v_1, \dots, v_m], \qquad L = \text{diag}(\lambda_1, \dots, \lambda_m)$$

form a deflating pair for P.

(4) Eigenvectors v_i to different eigenvalues λ_i are not necessarily linearly independent. However, the vectors $q(v_i, \lambda_i)$ are linearly independent if P_d is nonsingular.

Jordan chains I (finite eigenvalues)

Let $v_0, \ldots, v_{\ell-1} \in \mathbb{C}^n$, $v_0 \neq 0$, be a finite sequence of vectors. Let $P(\lambda) = \sum_{k=0}^{d} P_k \lambda^k$.
Then the following statements are equivalent for $\lambda_0 \in \mathbb{C}$. (proof by direct computation)

(i) $$\begin{bmatrix} P(\lambda_0) & & & \\ P'(\lambda_0) & P(\lambda_0) & & \\ \vdots & & \ddots & \\ \frac{P^{(\ell-1)]}(\lambda_0)}{(\ell-1)!} & \ldots & P'(\lambda_0) & P(\lambda_0) \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{\ell-1} \end{bmatrix} = 0 \quad \text{i.e.} \quad \sum_{k=0}^{m} \frac{P^{(m-k)}(\lambda_0)}{(m-k)!} v_k = 0,\ m = 0, \ldots, \ell-1.$$

(ii) The matrices $V = [v_0, \ldots, v_{\ell-1}]$ and $J = \begin{bmatrix} \lambda_0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_0 \end{bmatrix}$ form a deflating pair,

i.e. $0 = P_d V J^d + P_{d-1} V J^{d-1} + \ldots + P_1 V J + P_0 V$.

(iii) The vector polynomial $v(\lambda) := \sum_{k=0}^{\ell-1} (\lambda - \lambda_0)^k v_k$ (root polynomial) satisfies

$P(\lambda) v(\lambda) = u(\lambda) (\lambda - \lambda_0)^{\ell}$ for a vector polynomial $u(\lambda)$.

(iv) The functions $$x_m(t) = e^{\lambda_0 (t-t_0)} \left(\frac{(t-t_0)^m}{m!} v_0 + \frac{(t-t_0)^{m-1}}{(m-1)!} v_1 + \ldots + (t-t_0) v_{m-1} + v_m \right),$$

$m = 0, \ldots, \ell-1$, satisfy $P(\frac{d}{dt}) x_m(t) = 0$.

If these conditions are satisfied then the sequence $0 \neq v_0, \ldots, v_{\ell-1}$ is said to be a **Jordan chain** of length ℓ to the finite eigenvalue λ_0 of $P(\lambda)$.

A computation. We show $(i) \Leftrightarrow (iv)$ from the former slide. W.l.o.g. $t_0 = 0$.

Let $c_m := \sum_{k=0}^{m} \frac{P^{(m-k)}(\lambda_0)}{(m-k)!} v_k = \sum_{j=0}^{m} \frac{P^{(j)}(\lambda_0)}{j!} v_{m-j}$.

For any smooth function $f(t)$: $\left(\frac{d}{dt} - \lambda_0\right) \left(e^{\lambda_0 t} f(t)\right) = e^{\lambda_0 t} \frac{df}{dt}(t)$.

Thus, Taylor expansion of the polynomial P at λ_0 yields

$$P\left(\frac{d}{dt}\right) \left(e^{\lambda_0 t} f(t)\right) = \sum_{j=0}^{\infty} \frac{P^{(j)}(\lambda_0)}{j!} \left(\frac{d}{dt} - \lambda_0\right)^j \left(e^{\lambda_0 t} f(t)\right) = e^{\lambda_0 t} \sum_{j=0}^{\infty} \frac{P^{(j)}(\lambda_0)}{j!} \frac{d^j f}{dt^j}(t)$$

The sum is indeed finite. Now, for a vector polynomial f,

$$f(t) := \sum_{k=0}^{m} v_{m-k} \frac{t^k}{k!} \qquad \Rightarrow \qquad \frac{d^j f}{dt^j}(t) = \sum_{k=j}^{m} v_{m-k} \frac{t^{k-j}}{(k-j)!} = \sum_{s=0}^{m-j} v_{m-j-s} \frac{t^s}{s!}.$$

Thus,

$$P\left(\frac{d}{dt}\right) \left(e^{\lambda_0 t} f(t)\right) = e^{\lambda_0 t} \sum_{j=0}^{\infty} \frac{P^{(j)}(\lambda_0)}{j!} \sum_{s=0}^{m-j} v_{m-j-s} \frac{t^s}{s!} = e^{\lambda_0 t} \sum_{s=0}^{m} \underbrace{\left(\sum_{j=0}^{m-s} \frac{P^{(j)}(\lambda_0)}{j!} v_{(m-s)-j} \right)}_{c_{m-s}} \frac{t^s}{s!}.$$

Thus, $\quad P\left(\frac{d}{dt}\right) \left(e^{\lambda_0 t} f(t)\right) = 0 \qquad \Leftrightarrow \qquad c_s = 0, \;$ for $s = 0, \ldots, m.$ $\square$

If $v(\lambda) := \sum_{k=0}^{\ell-1} (\lambda - \lambda_0)^k v_k$ then $P(\lambda)\, v(\lambda) = \sum_s c_s \,(\lambda - \lambda_0)^s$. Hence, $(i) \Leftrightarrow (iii)$.

Canonical set of Jordan chains and local Smith form

For a regular $P(\lambda) \in \mathbb{C}^{n\times n}[\lambda]$ consider its **local Smith form** at the eigenvalue λ_0:

$$P(\lambda)\, \underbrace{[v_1(\lambda)\;\; v_2(\lambda)\;\; \dots\; v_n(\lambda)]}_{V(\lambda)} = \underbrace{[u_1(\lambda)\;\; u_2(\lambda)\;\; \dots\; u_n(\lambda)]}_{U(\lambda)} \begin{bmatrix} (\lambda-\lambda_0)^{\ell_1} & & \\ & \ddots & \\ & & (\lambda-\lambda_0)^{\ell_n} \end{bmatrix}.$$

with $V(\lambda), U(\lambda) \in \mathbb{C}^{n\times n}[\lambda]$, $\det(V(\lambda)) \neq 0 \neq \det(U(\lambda))$ and partial multiplicites ℓ_i.

The vector polynomials $v_i(\lambda), u_i(\lambda)$ satisfy $\quad P(\lambda)\, v_i(\lambda) = u_i(\lambda)\, (\lambda-\lambda_0)^{\ell_j}$.

Thus, if $\ell_i > 0$ then $v_i(\lambda) = \sum v_{i,k}(\lambda-\lambda_0)^k$ is a root polynomial for P, and its coeffcients

$$V_i = [v_{i,0}\; v_{i,1}\; \dots v_{i,\ell_j-1}]$$

form a Jordan chain to the eigenvalue λ_0. The collection of all these chains

$$V = [\dots\; V_i\; \dots], \quad \ell_i > 0$$

is said to be a **canonical set** of Jordan chains to the eigenvalue λ_0. Let

$$J = \mathrm{diag}(\dots\; J_i(\lambda_0)\; \dots), \quad \ell_i > 0$$

be a Jordan matrix with Jordan blocks $J_i(\lambda_0)$ of size ℓ_i. Then

$$0 = P_d V J^d + P_{d-1} V J^{d-1} + \dots + P_1 V J + P_0 V \quad \text{(deflating pair)}.$$

Homogenization, reverse polynomial and eigenvalue ∞

Consider the matrix polynomial $P(\lambda) = \sum_{k=0}^{d} P_k \lambda^k \in \mathbb{C}^{n\times n}[\lambda]$.

By setting $\lambda = \alpha/\beta$ and multiplying with β^d we obtain the homogenization of P:

$$P_h(\alpha,\beta) \;:=\; \beta^d P\left(\frac{\alpha}{\beta}\right) = \beta^d \sum_{k=0}^{d} P_k \left(\frac{\alpha}{\beta}\right)^k = \sum_{k=0}^{d} P_k\, \alpha^k \beta^{d-k}.$$

Observe that

$$P(\lambda) = P_h(\lambda, 1).$$

The **reverse of** P is defined as

$$P_r(\lambda) \;:=\; P_h(1,\lambda) = \sum_{k=0}^{d} P_k\, \lambda^{d-k} = \sum_{k=0}^{d} P_{d-k}\, \lambda^k.$$

Analogously the char. polynomial $\chi(\lambda) = \det(P(\lambda))$ is made homogeneous and reversed:

$$\chi_h(\alpha,\beta) := \beta^{nd} \chi\left(\frac{\alpha}{\beta}\right), \quad \chi_r(\lambda) = \chi_h(1,\lambda).$$

Definition of infinite eigenvalue. Let $m > 0$.

P has eigenvalue ∞ of alg. multiplicity m $:\Leftrightarrow$ P_r has eigenvalue 0 of alg. multiplicity m

$\Leftrightarrow$ 0 is a zero of multiplicity m of χ_r

$\Leftrightarrow$ χ has degree $nd - m$.

Jordan chains II (infinite eigenvalue)

Let $v_0, \ldots, v_{\ell-1} \in \mathbb{C}^n$, $v_0 \neq 0$, be a finite sequence of vectors.

Let $P(\lambda) = \sum_{k=0}^{d} P_k \lambda^k$ and $N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$.

Then the following statements are equivalent.

(i) $V = [v_0, v_1, \ldots, v_{\ell-1}]$ is a Jordan chain to the eigenvalue 0 of the reverse poly. P_r.

(ii) $0 = P_d V + P_{d-1} V N + \ldots + P_1 V N^{d-1} + P_0 V N^d$.

(iii)

$$\begin{bmatrix} 0 & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \\ -P_0 & -P_1 & \ldots & \ldots & -P_{d-1} \end{bmatrix} \underbrace{\begin{bmatrix} V N^{d-1} \\ V N^{d-2} \\ \vdots \\ V \end{bmatrix}}_{=:V_N} N = \begin{bmatrix} V N^{d-1} \\ V N^{d-2} \\ \vdots \\ P_d V \end{bmatrix}.$$

Definition: Any pair (V, N) with nilpotent N satisfying (2) is called a deflating pair to the eigenvalue ∞.

Structure of Quasi-Weierstrass form of the companion pencil

Consider the Quasi-Weiserstrass form of the companion pencil $C_P^\flat(\lambda)$ of the regular matrix polynomial $P(\lambda) \in \mathbb{C}^{n\times n}[\lambda]$,

$$\underbrace{\left(\lambda \begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & P_d \end{bmatrix} - \begin{bmatrix} 0 & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \\ -P_0 & -P_1 & \dots & \dots & -P_{d-1} \end{bmatrix}\right)}_{C_P^\flat(\lambda)} [V_F, V_\infty] = [U_F, U_\infty] \left(\lambda \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix}\right)$$

Then

$$V_F = \begin{bmatrix} V \\ V\,L \\ \vdots \\ V\,L^{d-1} \end{bmatrix}, \quad V_\infty = \begin{bmatrix} \hat{V}\,N^{d-1} \\ \hat{V}\,N^{d-2} \\ \vdots \\ \hat{V} \end{bmatrix},$$

and

$$\begin{aligned} 0 &= P_d\,V\,L^d + P_{d-1}\,V\,L^{d-1} + \ldots + P_1\,V\,L + P_0\,V, \\ 0 &= P_d\,\hat{V} + P_{d-1}\,\hat{V}\,N + \ldots + P_1\,\hat{V}\,N^{d-1} + P_0\,\hat{V}\,N^d. \end{aligned}$$

for some $V, \hat{V} \in \mathbb{C}^{n\times dn}$.

If L and N are in Jordan form then the columns of V and $\hat{V}$ form Jordan chains of P.

Solution formula for monic polynomials (same as in scalar case)

Let $\Phi_0, \ldots, \Phi_{d-1} : \mathbb{R} \to \mathbb{C}^{n\times n}$ be the solutions of the homogeneous initial value problem

$$\Phi_k^{(d)}(t)+P_{d-1}\,\Phi_k^{(d-1)}(t)+\ldots+P_1\,\dot{\Phi}_k(t)+P_0\,\Phi_k(t)=0, \qquad \Phi_k^{(j)}(0)=\delta_{jk}\,I, \quad j=0,\ldots,d-1$$

(Bohl functions). Then

$$x:\mathbb{R}\to\mathbb{C}^{n\times n}, \qquad x(t)=\sum_{k=0}^{d-1}\Phi_k(t-t_0)\,x_k+\int_{t_0}^{t}\Phi_{d-1}(t-s)\,f(s)\,ds$$

is the unique solution of the initial value problem

$$x^{(d)}(t)+P_{d-1}\,x^{(d-1)}(t)+\ldots+P_1\,\dot{x}(t)+P_0\,x(t)=f(t), \qquad x^{(k)}(t_0)=x_k.$$

The basis solutions Φ_k can be computed in the following way.

Step 1. Perform partial fraction expansion:

$$P(\lambda)^{-1}=\sum_{k=1}^{r}\sum_{j=1}^{m_k}\frac{A_{kj}}{(\lambda-\lambda_k)^j} \quad\Rightarrow\quad \Phi_{d-1}(t):=\sum_{k=1}^{r}e^{\lambda_k t}\sum_{j=1}^{m_k}A_{kj}\,\frac{t^{j-1}}{(j-1)!}.$$

Step 2. Set

$$\begin{aligned}
\Phi_{d-2} &:= P_{d-1}\,\Phi_{d-1}+\dot{\Phi}_{d-1},\\
\Phi_{n-3} &:= P_{d-2}\,\Phi_{n-1}+\dot{\Phi}_{d-2},\\
&\vdots\\
\Phi_0 &:= P_1\,\Phi_{d-1}+\dot{\Phi}_1.
\end{aligned}$$

Remarks on ODE of second order

The ODE/DAE of second order

$$M\,\ddot{x} + C\,\dot{x} + K\,x = f,$$

is equivalent to

(1) $\begin{bmatrix} \tilde{I} & 0 \\ 0 & M \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{I} \\ -K & -C \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}$ $\quad \tilde{I}$ nonsingular.

(2) $\begin{bmatrix} C & M \\ \tilde{I} & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -K & 0 \\ 0 & \tilde{I} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} f \\ 0 \end{bmatrix}$ $\quad \tilde{I}$ nonsingular,

(3) $\begin{bmatrix} M & C \\ 0 & M \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} 0 & -K \\ M & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} f \\ 0 \end{bmatrix},$

(4) $\begin{bmatrix} 0 & K \\ -M & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} K & 0 \\ C & K \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}.$

Many other equivalent formulations are possible. See next pages.

Remarks on ODE of second order

Consider again

$$M\,\ddot{x} + C\,\dot{x} + K\,x = f, \qquad (*)$$

If M, C, K are Hermitian positive semidefinite the associated **energy** is defined as

$$w(x,\dot{x}) = \frac{1}{2}(\,\dot{x}^* M \dot{x} + x^* K x).$$

It follows that

$$\frac{d}{dt} w(x,\dot{x}) = -\dot{x}^* C \dot{x} + \Re(f^* \dot{x}) \leq \Re(f^* \dot{x})$$

(This is a **dissipation inequality.** storage: $w(x,\dot{x})$. supply: $\Re(f^*\dot{x})$)

Veselic: If M and K are both positive definite factorize $\quad K = L_1^* L_1, \quad M = L_2^* L_2$ and define

$$y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} L_1\,x \\ L_2\,\dot{x} \end{bmatrix}, \quad \Omega := L_2^{-1} L_1, \quad D := L_2^{-1} C L_2^{-*}, \quad A := \begin{bmatrix} 0 & \Omega^* \\ -\Omega & -D \end{bmatrix}.$$

Then $(*)$ is equivalent to

$$\dot{y} = A\,y + \begin{bmatrix} 0 \\ L_2^{-1} f \end{bmatrix},$$

and we have

$$w(x,\dot{x}) = \frac{1}{2}\|y\|_2^2, \qquad f = 0 \;\Rightarrow\; \frac{d}{dt}\|y\|_2^2 = -y_2^* D y_2 \leq 0, \qquad \|e^{A\,t}\|_2 \leq 1.$$

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