GIAN course: Singular optimal control

Slide collection 3

Linear differential algebraic equations (DAE) with constant coefficients

Michael Karow, TU Berlin, December 2016

karow@math.tu-berlin.de

Subject of these notes

We consider equations of the form

$$E\,\dot{x}(t) = A\,x(t) + f(t), \qquad (\dot{x}\text{=first derivative of } x), \qquad (*)$$

where $E, A \in \mathbb{C}^{n \times n}$ and E may be singular.

In the latter case $(*)$ is not a pure differential equation anymore.

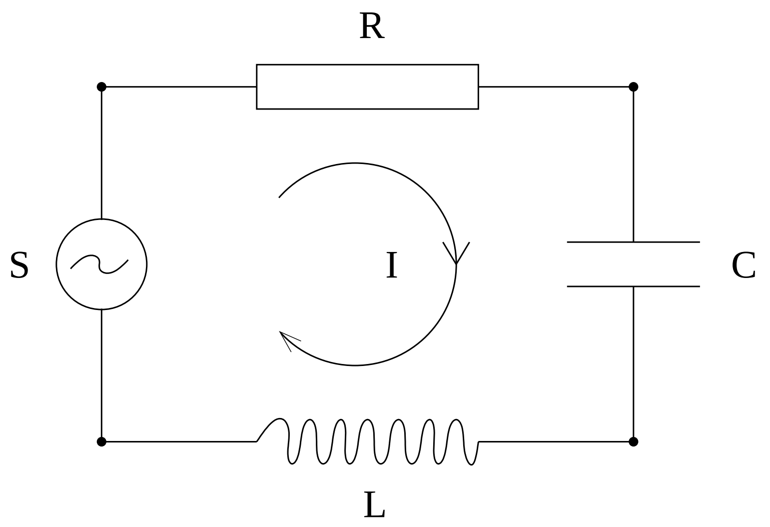
It is then called **Differential Algebraic Equation (DAE)**.

Systems of the form

$$E\,\dot{x} = A\,x + B\,u, \qquad y = C\,x + D\,u$$

are called **Descriptor Systems**.

Motivation 1: A simple descriptor system



Equations:

$I = C\,\dot{V}_C,$
$V_L = L\,\dot{I},$
$V_C = R\,I,$
$V_L + V_C + V_R + V_S = 0.$

All equations in matrix-vector form:

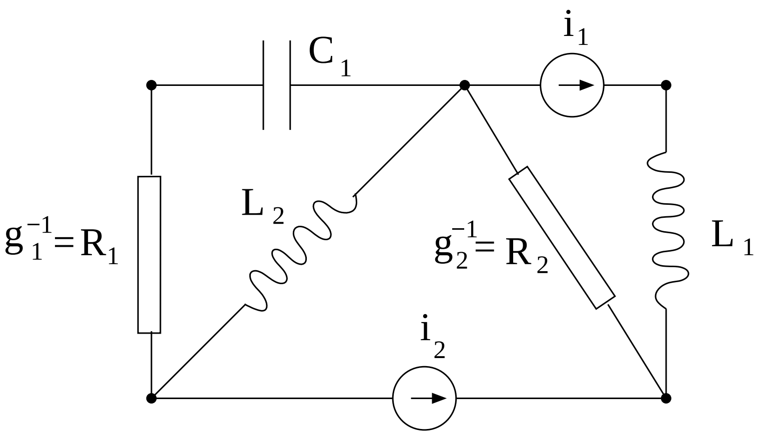
$$\begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{I}(t) \\ \dot{V}_L(t) \\ \dot{V}_C(t) \\ \dot{V}_R(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/C & 0 & 0 & 0 \\ -R & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} I(t) \\ V_L(t) \\ V_C(t) \\ V_R(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} V_S(t)$$

$\Rightarrow$

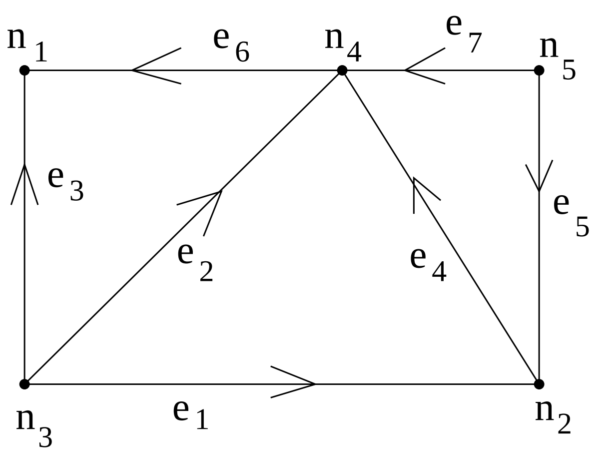
$$L\,\ddot{I}(t) + R\,\dot{I}(t) + C^{-1}\,I(t) = -\dot{V}_S(t).$$

Motivation 2: RLC networks with current sources

Electrical Network



Directed Graph



Incidence matrix of directed graph:

$$A_{in} = [a_{jk}] = \begin{array}{c} \begin{matrix} n_1 & n_2 & n_3 & n_4 & n_5 \end{matrix} \\ \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_5 \\ e_6 \\ e_7 \end{matrix}$$

$$a_{jk} = \begin{cases} 1 & \text{if } n_k \text{ is end point of } e_j, \\ -1 & \text{if } n_k \text{ is initial point of } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

RLC networks with current sources

$\imath =$ vector of currents
$v =$ vector of voltages
$p =$ vector of node potentials
$A_{in} =$ incidence matrix

Kirchhoff's laws: $\begin{array}{l} A_{in}^\top \imath = 0, \\ A_{in}\, p = v. \end{array}$

$$A_{in} = \begin{bmatrix} A_i \\ A_g \\ A_c \\ A_l \end{bmatrix}, \quad v = \begin{bmatrix} v_i \\ v_g \\ v_c \\ v_l \end{bmatrix}, \quad \imath = \begin{bmatrix} \imath_i \\ \imath_g \\ \imath_c \\ \imath_l \end{bmatrix}.$$

material laws:

$$\begin{array}{ll} \imath_i = -i(t) & \text{(current sources)} \\ \imath_g = \mathcal{G}\, v_g & \text{(conductances)} \\ \imath_c = \mathcal{C}\, \frac{d}{dt} v_c & \text{(capacitors)} \\ v_l = \mathcal{L}\, \frac{d}{dt} \imath_l & \text{(inductors)} \end{array}$$

$\Rightarrow$ **Descriptor System** (DAE with output):

$$\underbrace{\begin{bmatrix} \mathcal{L} & 0 \\ 0 & A_g^\top \mathcal{C} A_g \end{bmatrix}}_{E} \frac{d}{dt} \underbrace{\begin{bmatrix} \imath_l \\ p \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 & A_l \\ -A_l^\top & A_g^\top \mathcal{G} A_g \end{bmatrix}}_{-A} \underbrace{\begin{bmatrix} \imath_l \\ p \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ A_i^\top \end{bmatrix}}_{B} i(t), \qquad v_i(t) = B^\top x(t).$$

Laplace-trafo:

$$\widehat{v}_i(s) = \underbrace{B^\top (s\,E - A)^{-1} B}_{\text{transfer function}}\ \widehat{i}(s), \qquad \widehat{v}_i(s) := \int_0^\infty e^{-st}\, v(t)\, dt, \quad \widehat{i}(s) := \int_0^\infty e^{-st} i(t)\, dt, \quad s \in \mathbb{C}.$$

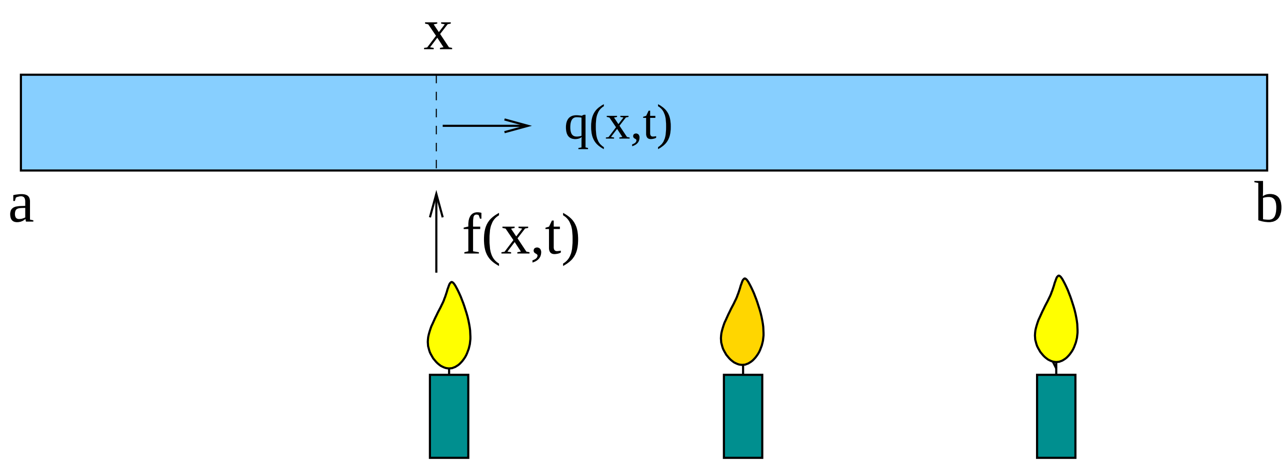
Motivation 2: Finite Element Method (toy example)

1-dimensional heat equation for temperature θ:

$$c(x)\,\frac{\partial}{\partial t}\theta(x,t) = \frac{d}{dx}\left(\lambda(x)\,\frac{\partial}{\partial x}\theta(x,t)\right) + f(x,t), \qquad \text{short:} \quad c\,\dot{\theta} = (\lambda\,\theta')' + f.$$

$c =$ heat capacity, $\lambda =$ thermal conductivity, $f =$ external heat source.

Heat flow: $$q(x,t) = -\lambda(x)\,\frac{\partial}{\partial x}\theta(x,t) \qquad \text{(Fourier's law)}$$



For unique solution a boundary value of θ or q must be prescribed at each end.

Galerkin's Method I

Heat equation: $c\,\dot\theta = (\lambda\,\theta')' + f$

Multiply with test func. $\psi(x)$ and integrate: $\int_a^b c\,\dot\theta\,\psi\;\,dx = \int_a^b [(\lambda\,\theta')' + f]\,\psi\,dx$

Partial integration with respect to x. (weak formulation) $\int_a^b c\,\dot\theta\,\psi\,dx = -\int_a^b \lambda\,\theta'\psi'\,dx + q_a\,\psi(a) - q_b\,\psi(b) + \int_a^b f\,\psi\,dx$

$q_a = -\lambda(a)\,\theta'(a,t)$ (heat flow at boundary)

In the weak formulation replace θ by approximation

$$\widehat{\theta}(x,t) := \textstyle\sum_{k=1}^{n} \theta_k(t)\,\phi_k(x),$$

where $\theta_k(t)$ are unknown coefficients and $\phi_1,\dots,\phi_n : [a,b] \to \mathbb{R}$ are given basis functions.

$$\sum_k \left(\int_a^b c\,\phi_k\,\psi\,dx\right)\dot\theta_k = -\sum_k \left(\int_a^b \lambda\,\phi_k'\psi'\,dx\right)\theta_k + q_a\,\psi(a) - q_b\,\psi(b) + \int_a^b f\,\psi\,dx$$

As test functions use $\psi = \phi_j$, $j = 1,\dots,n$, to get differential equations for the θ_k:

$$\sum_k \underbrace{\left(\int_a^b c\,\phi_k\,\phi_j\,dx\right)}_{=:m_{jk}}\dot\theta_k = -\sum_k \underbrace{\left(\int_a^b \lambda\,\phi_k'\phi_j'\,dx\right)}_{=:s_{jk}}\theta_k + q_a\,\phi_j(a) - q_b\,\phi_j(b) + \underbrace{\int_a^b f\,\phi_j\,dx}_{f_j}.$$

In matrix-vector form: $\boxed{M\,\underline{\dot\theta} = -S\,\underline{\theta} + q_a\,\underline{e}_a - q_b\,\underline{e}_b + \underline{f},}$

where

$$M := [m_{jk}], \quad S := [s_{jk}], \quad \underline{\theta} := [\theta_1\ \dots\ \theta_n]^\top, \quad \underline{e}_x := [\phi_1(x)\ \dots\ \phi_n(x)]^\top.$$

Galerkin's Method II

Result from slide before: $$M\,\underline{\dot\theta} = -S\,\underline{\theta} + q_a\,\underline{e}_a - q_b\,\underline{e}_b + \underline{f}, \qquad (*)$$

where

$$M := [m_{jk}], \quad S := [s_{jk}], \quad \underline{\theta} := [\theta_1 \ \ldots \ \theta_n]^\top, \quad \underline{e}_x := [\phi_1(x) \ \ldots \ \phi_n(x)]^\top.$$

and

$$\theta(x,t) \approx \widehat{\theta}(x,t) := \textstyle\sum_{k=1}^n \theta_k(t)\,\phi_k(x) = \underline{e}_x^\top\,\underline{\theta}(t)$$

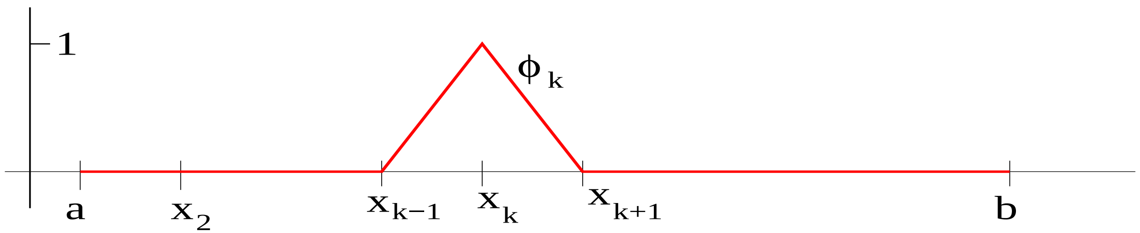
Suppose now, that boundary conditions

$$\theta(a,t) := \theta_a(t), \qquad q(b,t) = q_b(t)$$

are given. Then we obtain from $(*)$ the DAE

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{\dot\theta} \\ \dot q_a \end{bmatrix} = \begin{bmatrix} -S & \underline{e}_a \\ \underline{e}_a^\top & 0 \end{bmatrix} \begin{bmatrix} \underline{\theta} \\ q_a \end{bmatrix} + \begin{bmatrix} -q_b\,\underline{e}_b + \underline{f} \\ \theta_a \end{bmatrix}.$$

Finite element method: make matrices sparse by setting nodes x_k on $[a,b]$ and using the associated head functions as basis functions.



Motivation 4: Navier-Stokes equations

Equations for the velocity field v and the pressure p of an incompressible Newtonian flow with viscosity $\eta > 0$ and density ϱ:

(1) Balance of momentum:

$$\varrho\left(v \cdot \nabla v + \frac{d\,v}{dt}\right) = -\text{grad}\, p + \eta\,\Delta\, v$$

(2) Incompressibility: $\quad \text{div}\, v = 0.$

(3) Boundary conditions for v and p.

Linearization for slow flows: cancel the term $v \cdot \nabla v$.

Then (1) and (2) can be written formally as

$$\begin{bmatrix} \varrho\, I & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} \eta\,\Delta & -\text{grad} \\ \text{div} & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix}$$

Discretization leads to matrix-vector DAE of the same structure.

Linear DAE of first order with constant coefficients

We consider equations of the form

$$E\,\dot{x}(t) = A\,x(t) + f(t), \qquad (\dot{x}\text{=first derivative of } x) \qquad (*)$$

where $\mathbb{R} \ni t \longmapsto x(t), f(t) \in \mathbb{C}^n$ are functions and $A, E \in \mathbb{C}^{n\times n}$.

Let us consider 3 extreme cases.

<u>Case 1:</u> E is nonsingular. Then $(*)$ is equivalent to the ODE

$$\dot{x}(t) = E^{-1}A\,x(t) + E^{-1}\,f(t).$$

with solutions

$$x(t) = e^{E^{-1}A\,(t-t_0)}\,x(t_0) + \int_{t_0}^{t} e^{E^{-1}A\,(t-s)}\,E^{-1}f(s)\,ds.$$

<u>Case 2:</u> $E = 0$. Then $(*)$ is an algebraic equation.

<u>Case 3:</u> $A = I$, $0 \neq E =: N$ is nilpotent, i.e. $N^{\nu} = 0 \neq N^{\nu-1}$ for some $\nu \geq 1$.

See next page.

Lemma. The unique solution of the equation

$$N\,\dot{x}(t) = x(t) + f(t), \quad N^{\nu} = 0 \neq N^{\nu-1},\ \nu \geq 1, \qquad (*)$$

is

$$x(t) = -\sum_{k=0}^{\nu-1} N^k\, f^{(k)}(t) \qquad (f^{(k)} = k\text{-th derivative of } f).$$

Proof. This is obvious for $\nu = 1$. Let $\nu \geq 2$.

From $(*)$ it follows by differentiating and multiplying with powers of N,

$$\begin{aligned}
N\,\dot{x} &= x + f, \\
N^2\,\ddot{x} &= N\,\dot{x} + N\,\dot{f} \\
N^3\,x^{(3)} &= N^2\,\ddot{x} + N^2\,\ddot{f} \\
&\vdots \\
\underbrace{N^{\nu}}_{=0}\,x^{(\nu)} &= N^{\nu-1}\,x^{(\nu-1)} + N^{\nu-1}\,f^{(\nu-1)}.
\end{aligned}$$

Summing up:

$$N\,\dot{x} + N^2\,\ddot{x} + \ldots + N^{\nu-1}\,x^{(\nu-1)} = x + N\,\dot{x} + \ldots + N^{\nu-1}\,x^{(\nu-1)} + \sum_{k=0}^{\nu-1} N^k\, f^{(k)}(t).$$

This yields the result.

Eigenvector solutions

A scalar $\lambda_0 \in \mathbb{C}$ is said to be a **finite** eigenvalue of the pair $(E, A) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$ and its associated matrix pencil $\lambda\, E - A \in \mathbb{C}^{n\times n}[\lambda]$ if

$$\chi(\lambda_0) = 0, \qquad \text{where } \chi(\lambda) = \det(\lambda\, E - A) \text{ is the characteristic polynomial.}$$

To each finite eigenvalue λ_0 there exists a **nonzero** vector $v_0 \in \mathbb{C}^n$ such that

$$(\lambda_0\, E - A)\, v_0 = 0 \qquad (\Leftrightarrow \quad A\, v_0 = \lambda_0\, E\, v_0\ ).$$

Proposition. Let λ_k be finite eigenvalues such that

$$Av_k = \lambda_k\, Ev_k, \quad k = 1, \ldots, r.$$

Then for any choice of scalars α_k the function

$$x(t) := \sum_{k=1}^{r} \alpha_k\, e^{\lambda_k\,(t-t_0)}\, v_k$$

satisfies

$$E\,\dot{x}(t) = A\, x(t), \qquad x(t_0) = \sum_{k=1}^{r} \alpha_k\, v_k.$$

Verification.

$$E\,\dot{x}(t) \;=\; E\left(\sum_k \alpha_k\, \lambda_k\, e^{\lambda_k\,(t-t_0)}\, v_k\right) = A\left(\sum_k \alpha_k\, e^{\lambda_k\,(t-t_0)}\, v_k\right) \;=\; A\, x(t).$$

Regularity

The matrix pair $(E, A) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$ and its associated matrix pencil $\lambda\, E - A \in \mathbb{C}^{n\times n}[\lambda]$ are said to be **regular** if the **characteristic polynomial**

$$\chi(\lambda) = \det(\lambda\, E - A)$$

is not the zero polynomial.

On the next pages we consider only regular pairs (pencils) for the following reason.

Proposition. If (E, A) is not regular (i.e. singular) then the initial value problem

$$E\,\dot{x} = A\,x, \qquad x(t_0) = 0$$

has a nontrivial solution.

Proof. Since $\det(\lambda\, E - A) \equiv 0$ there are $n+1$ scalars λ_k and vectors $v_k \neq 0$ such that

$$(\lambda_k\, E - A)v_k = 0, \qquad k = 1, \ldots, n+1.$$

The vectors v_k are linearly dependent: $\sum_{k=1}^{n+1} \alpha_k\, v_k = 0, \quad (\alpha_1, \ldots, \alpha_{n+1}) \neq (0, \ldots, 0)$.

The function $\quad x(t) := \sum_{k=1}^{n+1} \alpha_k\, v_k\, e^{\lambda_k\,(t-t_0)} \quad$ has the required properties $\square$.

Example. The pair $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is singular.

The infinite eigenvalue

The matrix pair $(E, A) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}$ and its associated matrix pencil $\lambda E - A \in \mathbb{C}^{n \times n}[\lambda]$ are said to have the eigenvalue ∞ if

$$E \text{ is singular}$$

The associated eigenvectors are the nonzero vectors v with $Ev = 0$ (i.e. $v \in \text{Ker } E$).

Equivalence transformations

DAE:

$$E\,\dot{x}(t) = A\,x(t) + f(t). \qquad (*)$$

Two kinds of **equivalence transformations:**

(1) Multiplication of $(*)$ from the left with U^{-1}.

(2) Substitution: $x = V\,x_r$, where $x_r := V^{-1}x$ (new variable x_r).

Applying both transformations we obtain an equivalent DAE for the new variable x_r:

$$\boxed{\underbrace{(U^{-1}EV)}_{=:E_r}\,\dot{x}_r(t) = \underbrace{(U^{-1}AV)}_{:=A_r}\,x_r(t) + U^{-1}\,f(t).}$$

<u>Aim:</u> Find U, V such that E_r, A_r are as simple as possible (see canonical form later on).

The relations

$$E_r = U^{-1}EV, \qquad A_r = U^{-1}AV$$

can be written as pencil equation

$$U^{-1}(\lambda\,E - A)V = \lambda\,E_r - A_r \qquad \text{or} \qquad (\lambda\,E - A)V = U(\lambda\,E_r - A_r).$$

<u>Remark:</u> A pencil is a matrix polynomial of degree 1.
<u>Remark:</u> In the case $E = I$ one uses $U = V$.

Theorem. (Quasi-Weierstrass Form aka Standard Representation of a regular pencil)
Suppose the pencil $\lambda E - A \in \mathbb{C}^{n\times n}[\lambda]$ is regular, Then there exist nonsingular matrices $V = [V_F, V_\infty]$ and $U = [U_F, U_\infty]$, with $V_\infty, U_\infty \in \mathbb{C}^{n\times d}$ as well as $\Lambda \in \mathbb{C}^{(n-d)\times(n-d)}$ and a nilpotent matrix $N \in \mathbb{C}^{d\times d}$ s.t.

$$(\lambda E - A)[V_F, V_\infty] = [U_F, U_\infty]\left(\lambda\begin{bmatrix} I & 0 \\ 0 & N\end{bmatrix} - \begin{bmatrix}\Lambda & 0 \\ 0 & I\end{bmatrix}\right). \qquad (*)$$

If E is nonsingular (resp. $E = 0$) the matrices V_∞, U_∞, N (resp. V_F, U_F, Λ) are not present.

Corollary. For arbitrary functions $t \mapsto x(t), f(t) \in \mathbb{C}^n$ let

$$\begin{bmatrix} x_F \\ x_\infty \end{bmatrix} := [V_F, V_\infty]^{-1}x, \quad \begin{bmatrix} f_F \\ f_\infty \end{bmatrix} := [U_F, U_\infty]^{-1}f, \quad \text{i.e.} \quad x = V_F\, x_F + V_\infty\, x_\infty, \quad f = U_F\, f_F + U_\infty\, f_\infty.$$

Then

$$E\dot{x} = Ax + f \quad \Leftrightarrow \quad (\, \dot{x}_F = \Lambda\, x_F + f_F, \quad N\,\dot{x}_\infty = f_\infty\,)$$

$$\Leftrightarrow \quad x(t) = V_F\left(e^{\Lambda\,(t-t_0)}\,x_F(t_0) + \int_{t_0}^{t} e^{\Lambda\,(t-s)}\,f_F(s)\,ds\right) + V_\infty\left(-\sum_{k=0}^{\nu-1} N^k f_\infty^{(k)}(t)\right)$$

Remarks.

(i) $(*)$ is said to be a Weierstrass form if Λ and N are both in Jordan canonical form.

(ii) $(*)$ is equivalent to $[U_F, U_\infty] = [E\,V_F, A\,V_\infty]$, $AV_F = EV_F\Lambda$, $EV_\infty = AV_\infty N$.

(iii) The solutions of $E\,\dot{x} = A\,x$ take values in the subspace $\mathcal{V}_F := \operatorname{im} V_F$.

(iv) The initial value problem $E\dot{x} = Ax + f$, $x(t_0) = x_0$ has a (unique) solution iff

$$x_0 + V_\infty \textstyle\sum_{k=0}^{\nu-1} N^k f_\infty^{(k)}(t_0) \in \mathcal{V}_F \qquad \textbf{(consistency condition)}.$$

Proof (Campbell). Choose $\lambda_0 \in \mathbb{C} \setminus \{0\}$ such that $\lambda_0 E - A$ is nonsingular. Define

$$\left.\begin{array}{l} E_0 := E(\lambda_0 E - A)^{-1} \\ A_0 := A(\lambda_0 E - A)^{-1} \end{array}\right\} \quad \Rightarrow \quad \lambda_0 E_0 - A_0 = I \quad \Rightarrow \quad A_0 = \lambda_0 E_0 - I.$$

Now, block diagonalize E_0:

$$E_0 = V_0 \begin{bmatrix} \Lambda_0 & 0 \\ 0 & N_0 \end{bmatrix} V_0^{-1}, \quad \Lambda_0 \text{ nonsingular}, \quad N_0 \text{ nilpotent, } N_0^\nu = 0.$$

Then

$$A_0 = V_0 \begin{bmatrix} \lambda_0 \Lambda_0 - I & 0 \\ 0 & \lambda_0 N_0 - I \end{bmatrix} V_0^{-1}.$$

The matrix $\lambda_0 N_0 - I$ is nonsingular since $(\lambda_0 N_0 - I)\overbrace{(-\sum_{k=0}^{\nu-1} \lambda_0^{-k} N_0^k)}^{(\lambda_0 N_0 - I)^{-1}} = I$. Now,

$$E\overbrace{(\lambda_0 E - A)^{-1} V_0}^{V} = E_0 V_0 = V_0 \begin{bmatrix} \Lambda_0 & 0 \\ 0 & N_0 \end{bmatrix} = \overbrace{V_0 \begin{bmatrix} \Lambda_0 & 0 \\ 0 & \lambda_0 N_0 - I \end{bmatrix}}^{U} \begin{bmatrix} I & 0 \\ 0 & \underbrace{(\lambda_0 N_0 - I)^{-1} N_0}_{N} \end{bmatrix}$$

$$A\underbrace{(\lambda_0 E - A)^{-1} V_0}_{V} = A_0 V_0 = \underbrace{V_0 \begin{bmatrix} \Lambda_0 & 0 \\ 0 & \lambda_0 N_0 - I \end{bmatrix}}_{U} \begin{bmatrix} \overbrace{\Lambda_0^{-1}(\lambda_0 \Lambda_0 - I)}^{\Lambda} & 0 \\ 0 & I \end{bmatrix}$$

Since N_0 and $(\lambda_0 N_0 - I)^{-1}$ commute, $N^\nu = (\lambda_0 N_0 - I)^{-\nu} N_0^\nu = 0$.

Proposition (uniqueness of standard rep. up to similarity). Let

$$(\lambda E - A)[V_F, V_\infty] \;=\; [U_F, U_\infty]\left(\lambda \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix}\right) \qquad (*)$$

be a standard representation of the regular pencil $\lambda E - A$.

Then all standard representations are obtained from it

via a similarity transformation. More precisely,

$$(\lambda E - A)[\tilde{V}_F, \tilde{V}_\infty] \;=\; [\tilde{U}_F, \tilde{U}_\infty]\left(\lambda \begin{bmatrix} I & 0 \\ 0 & \tilde{N} \end{bmatrix} - \begin{bmatrix} \tilde{\Lambda} & 0 \\ 0 & I \end{bmatrix}\right). \qquad (**)$$

is a standard representation if and only if

$$\begin{aligned} \tilde{\Lambda} &= P\Lambda P^{-1}, & \left[\tilde{V}_F, \tilde{V}_\infty\right] &= \left[V_F P, \tilde{V}_\infty Q\right], \\ \tilde{N} &= QNQ^{-1}, & \left[\tilde{U}_F, \tilde{U}_\infty\right] &= \left[U_F P^{-1}, \tilde{U}_\infty Q^{-1}\right]. \end{aligned}$$

for invertible P, Q of appropriate size.

Consequences: In a standard representation

- the matrices $V_F, V_\infty, U_F, U_\infty$ are not unique but the spaces $\mathcal{V}_F = \operatorname{im} V_F$, $\mathcal{V}_\infty = \operatorname{im}\, V_\infty$, $\operatorname{im} U_F = A\,\mathcal{V}_F$, $\operatorname{im} U_\infty = E\,\mathcal{V}_\infty$ are.
- the Jordan canonical forms of Λ and N are unique, in particular $\operatorname{index}(A, E) := \operatorname{index}(N)$ is well defined. We set $\operatorname{index}(A, E) := 0$ if E is nonsingular (i.e. N is not present).

Proof. Let $M, \tilde{M}$ be partitioned nonsingular matrices such that

$$\begin{bmatrix}\tilde{V}_F & \tilde{V}_\infty\end{bmatrix} = \begin{bmatrix}V_F & V_\infty\end{bmatrix} \underbrace{\begin{bmatrix}P & R\\ S & Q\end{bmatrix}}_{M}, \quad \begin{bmatrix}U_F & U_\infty\end{bmatrix} = \begin{bmatrix}\tilde{U}_F & \tilde{U}_\infty\end{bmatrix} \underbrace{\begin{bmatrix}\tilde{P} & \tilde{R}\\ \tilde{S} & \tilde{Q}\end{bmatrix}}_{\tilde{M}}.$$

Combining $(*)$ and $(**)$ it follows that

$$\begin{bmatrix}\tilde{P} & \tilde{R}\\ \tilde{S} & \tilde{Q}\end{bmatrix}\left(\lambda\begin{bmatrix}I & 0\\ 0 & N\end{bmatrix} - \begin{bmatrix}\Lambda & 0\\ 0 & I\end{bmatrix}\right) = \left(\lambda\begin{bmatrix}I & 0\\ 0 & \tilde{N}\end{bmatrix} - \begin{bmatrix}\tilde{\Lambda} & 0\\ 0 & I\end{bmatrix}\right)\begin{bmatrix}P & R\\ S & Q\end{bmatrix}$$

This yields

$$\begin{aligned} &\tilde{P} = P, \quad \tilde{R}N = R, \quad \tilde{S} = \tilde{N}S, \quad \tilde{Q}N = \tilde{N}Q,\\ &\tilde{P}\Lambda = \tilde{\Lambda}P, \quad \tilde{R} = \tilde{\Lambda}R, \quad \tilde{S}\Lambda = S, \quad \tilde{Q} = Q. \end{aligned}$$

Hence,

$$S = \tilde{S}\Lambda = \tilde{N}S\Lambda.$$

which implies

$$S = \tilde{N}S\Lambda = \tilde{N}(\tilde{N}S\Lambda)\Lambda = \tilde{N}^2S\Lambda^2 = \ldots = \tilde{N}^\nu S\Lambda^\nu = 0 \quad \text{and}$$

$$\tilde{S} = \tilde{N}S = 0, \quad \tilde{R} = \tilde{\Lambda}S = 0, \quad R = \tilde{R}N = 0.$$

Thus,

$$\tilde{M} = M = \begin{bmatrix}P & 0\\ 0 & Q\end{bmatrix}.$$

Won sequences

For the regular pencil $\lambda E - A$, define two sequences of subspaces (Won sequences) by

$$\mathcal{V}_F^0 := \mathbb{C}^n, \quad \mathcal{V}_F^{k+1} := A^{-1}E\mathcal{V}_F^k = \{v \in \mathbb{C}^n \,|\, \exists z \in \mathcal{V}_F^k: \ Av = Ez\}.$$

$$\mathcal{V}_\infty^0 := \{0\}, \quad \mathcal{V}_\infty^{k+1} := E^{-1}A\mathcal{V}_\infty^k = \{v \in \mathbb{C}^n \,|\, \exists z \in \mathcal{V}_\infty^k: \ Ev = Az\}.$$

Theorem (Berger, Ilchmann, Trenn, 2012).

Let $\nu = \text{index}(A, E)$. The Won sequences satisfy

(i) $\mathcal{V}_F^0 \supset \mathcal{V}_F^1 \supset \ldots \supset \mathcal{V}_F^\nu = \mathcal{V}_F^{\nu+1} =: \mathcal{V}_F$

(ii) $\mathcal{V}_\infty^0 \subset \mathcal{V}_\infty^1 \subset \ldots \subset \mathcal{V}_\infty^\nu = \mathcal{V}_\infty^{\nu+1} =: \mathcal{V}_\infty$

(iii) $\mathcal{V}_F \oplus \mathcal{V}_\infty = \mathbb{C}^n$.

Let V_F (resp. V_∞) be a matrix whose columns form a basis of $\mathcal{V}_F$ (resp. $\mathcal{V}_\infty$).

Then there exist Λ, N such that

$$(\lambda E - A)[V_F, V_\infty] = [EV_F, AV_\infty]\left(\lambda \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix}\right). \qquad (*)$$

is a Quasi-Weierstrass form (this follows from the definition of the spaces).

The fundamental solution

Quasi-Weierstrass form:

$$(\lambda E - A)V = U\left(\lambda\begin{bmatrix} I & 0 \\ 0 & N\end{bmatrix} - \begin{bmatrix} \Lambda & 0 \\ 0 & I\end{bmatrix}\right) \qquad \Leftrightarrow \qquad E = U\begin{bmatrix} I & 0 \\ 0 & N\end{bmatrix}V^{-1},\;\; A = U\begin{bmatrix} \Lambda & 0 \\ 0 & I\end{bmatrix}V^{-1}.$$

Define

$$\Phi(t) := V\begin{bmatrix} e^{\Lambda t} & 0 \\ 0 & 0\end{bmatrix}U^{-1}, \qquad \Phi_k := V\begin{bmatrix} 0 & 0 \\ 0 & N^k\end{bmatrix}U^{-1}.$$

Then all solutions of $E\dot{x} = Ax + f$ can be written as

$$\boxed{x(t) \;=\; \Phi(t)\,E\,x_0 + \int_0^t \Phi(t-s)\,f(s)\,ds - \sum_{k=0}^{\nu-1}\Phi_k\,f^{(k)}(t) \qquad x_0 \in \mathbb{C}^n \text{ arbitrary.}}$$

Observe, that in general $x(0) \neq x_0$ but

$$\boxed{x(0) = \Phi(0)E\,x_0 - \sum_{k=0}^{\nu-1}\Phi_k\,f^{(k)}(0)}$$

Note further that $P_F := \Phi(0)E = V\begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}V^{-1}$ is the projection onto $\mathcal{V}_F$ along $\mathcal{V}_\infty$.

Eigenvalues and eigenvectors

Definitions.

A vector $v \in \mathbb{C}^n \setminus \{0\}$ is called an eigenvector of the pair $(E, A) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$ (or the pencil $\lambda E - A$) to the finite eigenvalue $\lambda_0 \in \mathbb{C}$ if

$$Av = \lambda_0 \, Ev \quad (\Leftrightarrow \; (\lambda_0 \, E - A)v = 0 \;)$$

Thus, in the regular case, the distinct finite eigenvalues λ_i are the zeros of the polynomial

$$\chi(\lambda) = \det(\lambda \, E - A) = c \prod_i (\lambda - \lambda_i)^{m_i}, \qquad c \neq 0.$$

The exponent m_i is the algebraic multiplicity of λ_i.

If $m_\infty := n - \sum_i m_i > 0$ then (A, E) is said to have the eigenvalue ∞ of alg. mult. m_∞.

Remarks: (i) If $\chi(\lambda) \equiv c \neq 0$ then there are no finite eigenvalues.
(ii) If $Av_i = \lambda_i \, v_i$ then for any scalars α_i the function

$$x(t) := \sum_i \alpha_i \, v_i \, e^{\lambda_i (t - t_0)}$$

solves the initial value problem

$$E \, \dot{x} = Ax, \qquad x(t_0) = \sum_i \alpha_i \, v_i.$$

Jordan chains I (finite eigenvalues)

Let $v_0, \ldots, v_{\ell-1} \in \mathbb{C}^n$, $v_0 \neq 0$, be a finite sequence of vectors.
Then the following statements are equivalent for $\lambda_0 \in \mathbb{C}$. (proof by direct computation)

(i) $Av_0 = \lambda_0 E v_0$ and $Av_k = E(\lambda_0 v_k + v_{k-1})$ for $k = 1, \ldots, \ell - 1$.

(ii) The matrix $V = [v_0, \ldots, v_{\ell-1}]$ satisfies $A V = E V J$, where $J = \begin{bmatrix} \lambda_0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_0 \end{bmatrix}$.

[Pencil equation $(\lambda E - A)V = EV(\lambda I - J)$.]

(iii) The vector polynomial $v(\lambda) := \sum_{k=0}^{\ell-1} v_k (\lambda - \lambda_0)^k$ (root polynomial) satisfies

$$(\lambda E - A) v(\lambda) = E v_{\ell-1} (\lambda - \lambda_0)^\ell.$$

(iv) The functions

$$x_p(t) = e^{\lambda_0 (t-t_0)} \left(\frac{(t-t_0)^p}{p!} v_0 + \frac{(t-t_0)^{p-1}}{(p-1)!} v_1 + \ldots + (t-t_0) v_{p-1} + v_p \right),$$

$p = 0, \ldots, \ell - 1$, satisfy $E \dot{x}_p = A x_p$.

If these conditions are satisfied then the sequence $0 \neq v_0, \ldots, v_{\ell-1}$ is said to be a **Jordan chain** of length ℓ to the finite eigenvalue λ of the pair (E, A).

Jordan chains II (infinite eigenvalue)

Let $v_0, \dots, v_{\ell-1} \in \mathbb{C}^n$, $v_0 \neq 0$, be a finite sequence of vectors.
Then the following statements are equivalent. (proof by direct computation)

(i) $E\, v_0 = 0$ and $E\, v_k = A\, v_{k-1}$ for $k = 1, \dots, \ell - 1$.

(ii) The matrix $V = [v_0, \dots, v_{\ell-1}]$ satisfies $E\,V = A\,V\,N$, where $N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$.

[Pencil equation $(\lambda\, E - A)V = AV(\lambda\, N - I)$.]

(iii) The vector polynomial $v(\lambda) := \sum_{k=0}^{\ell-1} v_k\, \lambda^k$ (root polynomial) satisfies

$$(\lambda\, A - E)v(\lambda) = A\, v_{\ell-1}\, \lambda^\ell.$$

(iv) For every ℓ-times differentiable function $t \mapsto \phi(t) \in \mathbb{C}$ the functions

$$x_p(t) = -(v_0\, \phi^{(p)} + v_1\, \phi^{(p-1)} + \dots + \dot{\phi}\, v_{p-1} + \phi\, v_p), \qquad p = 0, \dots, \ell - 1,$$

satisfy $E\, \dot{x}_p = A(x_p + \phi\, v_p)$.

If these conditions are satisfied then the sequence $0 \neq v_0, \dots, v_{\ell-1}$ is said to be a **Jordan chain** of length ℓ to the finite eigenvalue ∞ of the pair (E, A).

Connection between Jordan chains and Weierstrass Form

(i) The elements of a Jordan chain to a finite eigenvalue are contained in $\mathcal{V}_F$.

(ii) The elements of a Jordan chain to the value ∞ are contained in $\mathcal{V}_\infty$.

(iii) The space $\mathcal{V}_F$ has a basis consisting of Jordan chains to the finite eigenvalues.
The space $\mathcal{V}_\infty$ has a basis consisting of Jordan chains to the eigenvalue ∞.
More precisely, let

$$(\lambda E - A)[V_F, V_\infty] = [U_F, U_\infty] \left(\lambda \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} \right)$$

be a Weierstrass form, i.e. Λ and N are in Jordan canonical form.
Then the columns of V_F form Jordan chains to the finite eigenvalues,
the columns of V_∞ form Jordan chains to the eigenvalue ∞.

Special case: index-1-system. Let

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad E_{11}, A_{11} \in \mathbb{C}^{d\times d}, A_{22} \in \mathbb{C}^{(n-d)\times(n-d)}.$$

If E_{11} and A_{22} are nonsingular, then (E, A) has index 1.

Proof. We have

$$\left(\lambda \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{bmatrix}$$

$$= \begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \left(\lambda \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21}) & 0 \\ 0 & A_{22} \end{bmatrix}\right)$$

$$= \begin{bmatrix} E_{11} & A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix} \left(\lambda \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} E_{11}^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21}) & 0 \\ 0 & I \end{bmatrix}\right)$$

Remark. The inhomogeneous DAE system associated with A, E is

$$E_{11}\dot{x}_1 = A_{11}\,x_1 + A_{12}\,x_2 + f_1, \quad 0 = A_{21}\,x_1 + A_{22}\,x_2 + f_2.$$

If A_{22} is nonsingular, we can solve for x_2 and insert the result into the ODE. Thus,

$$x_2 = -A_{22}^{-1}(A_{21}\,x_1 + f_2), \quad E_{11}\,\dot{x}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})\,x_1 + f_1 - A_{12}A_{22}^{-1}f_2.$$

Remark. It can be shown that (E, A) above has index > 1 if $\det(E_{11}) \neq 0 = \det(A_{22})$.

Regularization (index reduction) by feedback

Consider the DAE

$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u.$$

Define v by

$$u = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v.$$

Then

$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} + B_1 F_1 & A_{12} + B_1 F_2 \\ A_{21} + B_2 F_1 & A_{22} + B_2 F_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} v$$

This DAE has index 1 if E_{11} is nonsingular and F_2 is chosen such that $A_{22} + B_2 F_2$ is nonsingular.

Special Case: index-3-system.

The linear DAE of second order (mechanical system mit constraint)

$$M\ddot{x} + C\,\dot{x} + K\,x + G^*\mu = f \qquad Gx = g$$

(M, C, K mass, damping, stiffness matrix, f external force, $G^*\mu$ constraint force) can be linearized as

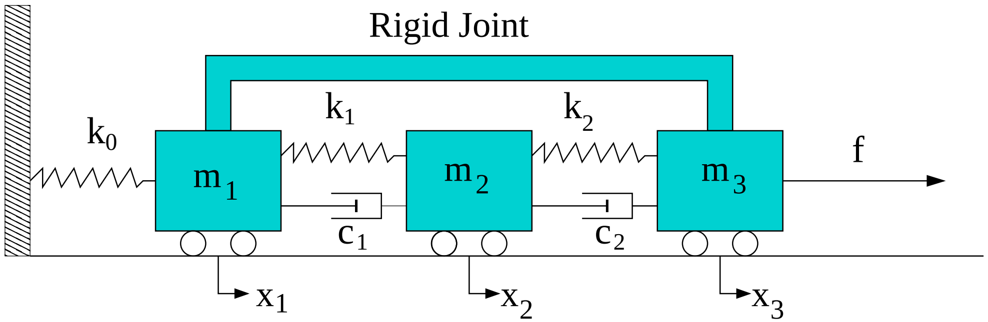
$$\underbrace{\begin{bmatrix} 0 & I & 0 \\ 0 & 0 & M \\ 0 & 0 & 0 \end{bmatrix}}_{E} \begin{bmatrix} \dot{\mu} \\ \dot{x} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & I \\ -G^* & -K & -C \\ 0 & -G & 0 \end{bmatrix}}_{A} \begin{bmatrix} \mu \\ x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f \\ g \end{bmatrix}. \qquad \text{(Note: } v = \dot{x}\text{)}$$

If M and $\begin{bmatrix} G^* & K \\ 0 & G \end{bmatrix}$ are nonsingular, then index(E) = index$(E, A) = 3$.

Proof. Exercise.

Example on the next page.

Example of a linear mechanical system with geometric constraint



Equilibrium position: $x_1 = x_2 = x_3 = 0.$

Geometric constraint: $x_1 - x_3 = const = 0.$

Equations of motion:

$$
\begin{aligned}
m_1 \ddot{x}_1 &= -k_0 x_1 + k_1 (x_2 - x_1) + c_1 (\dot{x}_2 - \dot{x}_1) + \mu, \\
m_2 \ddot{x}_2 &= -k_1 (x_2 - x_1) - c_1 (\dot{x}_2 - \dot{x}_1) + k_2 (x_3 - x_2) + c_2 (\dot{x}_3 - \dot{x}_2), \\
m_3 \ddot{x}_3 &= -k_2 (x_3 - x_2) - c_2 (\dot{x}_3 - \dot{x}_2) + f - \mu.
\end{aligned}
$$

μ is constraint force. Equations of motion in matrix-vector form:

$$
\begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_2 \end{bmatrix}
\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}
=
\begin{bmatrix} -k_0 - k_1 & k_1 & \\ k_1 & -k_1 - k_2 & k_2 \\ & k_2 & -k_2 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
+
\begin{bmatrix} -c_1 & c_1 & \\ c_1 & -c_1 - c_2 & c_2 \\ & c_2 & -c_2 \end{bmatrix}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}
$$

$$
+ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \mu + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} f.
$$

The Kronecker canonical form of an arbitrary (not necessarily square) pencil

Let $E, A \in \mathbb{C}^{n\times m}$. Then there exist nonsingular U, V such that

$$\begin{aligned} E &= U \operatorname{diag}(E_1, \ldots, E_k, \ldots, E_r)\, V^{-1}, \\ A &= U \operatorname{diag}(A_1, \ldots, A_k, \ldots, A_r)\, V^{-1}, \end{aligned} \qquad (*)$$

where each pair E_k, A_k is of one of the following types.

(1) $E_k = I,\ A_k = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$, (2) $E_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$, $A_k = I$,

(3) $E_k = \begin{bmatrix} 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}$, $A_k = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix}$, (4) transpose of (3).

(5) $E_k = A_k =$ zero matrix of size $p \times q$.

Equations $(*)$ can be written as **pencil equation** $U^{-1}(\lambda E - A)\, V = \bigoplus_k (\lambda E_k - A_k)$.

The blocks (E_k, A_k) are **unique** up to ordering. The blocks (3), (4), (5) are called singular.

Pairs of type (5) can be considered as block diagonal matrices consisting of blocks of type (3),(4) with no columns or no rows.

Remarks on numerical methods.

This would require a lecture series on its own. Implicit methods are necessary.

Let x_k be approximation of exact solution $x(t_k)$. Let $\Delta t = t_{k+1} - t_k$ be the step width.

Most primitive: **implicit Euler method**, i.e.

$$E\,\dot{x} = A\,x + f \quad \rightsquigarrow \quad E\,\frac{x_{k+1} - x_k}{\Delta t} = A\,x_{k+1} + f_{k+1} \quad \Rightarrow \quad (E - \Delta t\,A)x_{k+1} = E\,x_k + \Delta t\,f_{k+1}.$$

More sophisticated: **BDF(2)**, i.e.

$$E\,\dot{x} = A\,x + f \qquad \rightsquigarrow \qquad E\,\frac{\frac{3}{2}\,x_{k+1} - 2\,x_k + \frac{1}{2}\,x_{k-1}}{\Delta t} = A\,x_{k+1} + f_{k+1}$$

Quick overview on numerical methods in **book of Kunkel and Mehrmann.**

MATLAB `ode15i` solves DAE of index 1.

For eigenvalues and quasi-Weierstrass form use MATLAB commands `eig` or `qz`.

An algorithm for computation of Kronecker form can be found in
Paul Van Dooren: The computation of Kronecker's canonical form of a singular pencil. (LAA 27, 1979)

Literature (only a view references out of hundreds)

Liyi Dai. Singular Control Systems (1989)

Stephen L. Campbell. Singular Systems of Differential Equations.
Available from Campbell's web page: `http://www4.ncsu.edu/ slc/`

P. Kunkel, V. Mehrmann.
Differential-algebraic Equations: Analysis and Numerical Solution. (2006)

Theory of time varying linear systems: $E(t)\,\dot{x}(t) = A(t)\,x(t) + f(t)$.

Quick overview of numerical methods.

Ricardo Riaza.
Differential-algebraic Systems: Analytical Aspects and Circuit Applications.(2008)