GIAN course: Singular optimal control

Slide collection 1

Introduction

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The lecturer

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Research Group
Modeling, Numerical Analysis,
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Research interests:
Linear Algebra, Control theory

Teaching:
Numerical Analysis for engineers
(including modeling)

Linear Systems

Classical linear system with state x, input u and output y:

$$\dot{x}(t) = A\,x(t) + B\,u(t), \qquad (*)$$
$$y(t) = C\,x(t) + D\,u(t)$$

Descriptor system (singular system):

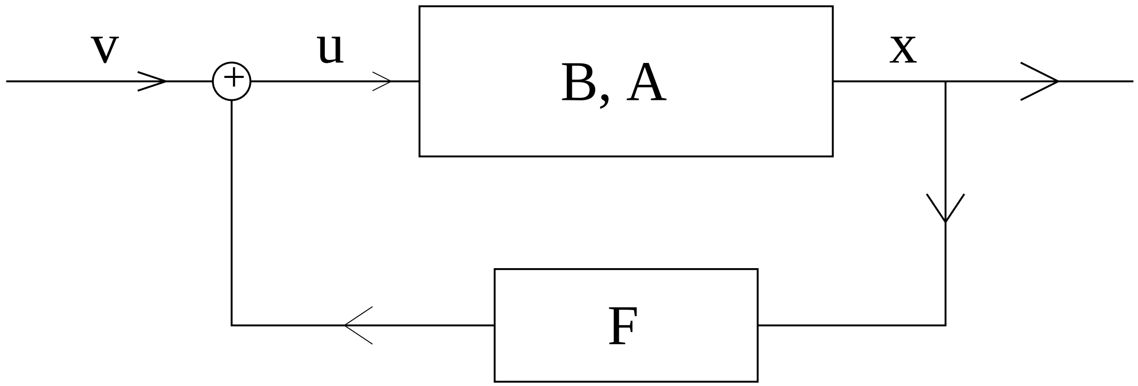
$$E\,\dot{x}(t) = A\,x(t) + B\,u(t), \qquad (**)$$
$$y(t) = C\,x(t) + D\,u(t)$$

$(*)$ is an ordinary differential equation (ODE).

$(**)$ if E is singular, then $(**)$ is a differential algebraic equation (DAE).

Why E? (simple example soon).

Changing system properties by feedback



System: $E\,\dot{x}(t) = A\,x(t) + B\,u(t)$

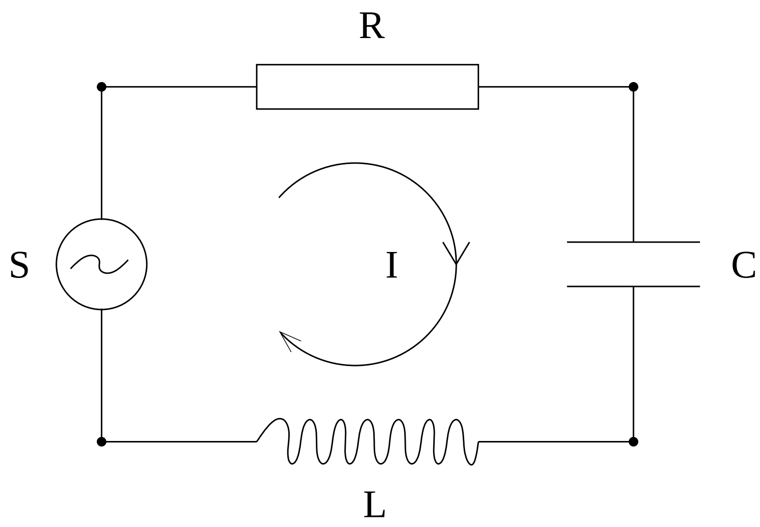
State feedback: $u(t) = F\,x(t) + v(t)$

Closed Loop: $E\,\dot{x}(t) = (A + BF)\,x(t) + B\,v(t)$

Stabilization: Find F such that (for $v(t) \equiv 0$),

$$\lim_{t\to\infty} x(t) = 0.$$

A simple descriptor system (RLC-circuit)



Equations:

$$I = C\,\dot{V}_C,$$
$$V_L = L\,\dot{I},$$
$$V_C = R\,I,$$
$$V_L + V_C + V_R + V_S = 0.$$

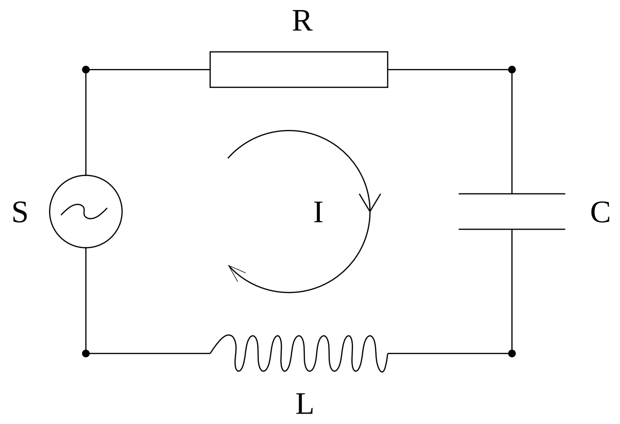
All equations in matrix-vector form:

$$\begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{I}(t) \\ \dot{V}_L(t) \\ \dot{V}_C(t) \\ \dot{V}_R(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/C & 0 & 0 & 0 \\ -R & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} I(t) \\ V_L(t) \\ V_C(t) \\ V_R(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} V_S(t)$$

$\Rightarrow$

$$L\,\ddot{I}(t) + R\,\dot{I}(t) + C^{-1}\,I(t) = -\dot{V}_S(t).$$

Dissipation inequality



Energy stored in the circuit (**storage function**):

$$w(t) = \frac{C}{2} V_C(t)^2 + \frac{L}{2} I(t)^2$$

Energy supplied to the circuit (**supply function**):

$$s(t) = -I(t) V_S(t)$$

For $t_1 < t_2$ we have the disspation inequality:

$$w(t_2) = w(t_1) - \int_{t_1}^{t_2} R\, I(t)^2\, dt + \int_{t_1}^{t_2} s(t)\, dt \leq w(t_1) + \int_{t_1}^{t_2} s(t)\, dt.$$

Linear quadratic optimal control

Problem: Minimize the cost functional

$$J(x,u) = \int_{t_0}^{T} \begin{bmatrix} x \\ u \end{bmatrix}^* \underbrace{\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}}_{\text{Hermitian}} \begin{bmatrix} x \\ u \end{bmatrix} dt = \int_{t_0}^{T} x^*Qx + 2\,\Re(u^*Sx) + u^*Ru\, dt.$$

subject to

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0.$$

Related question: is there a a storage function w for which J is the supply?

Partial answer: If a storage function w exists it is of the form

$$w(x) = x^*Px,$$

where P satisfies a **Linear Matrix Inequality** or a **Riccati Equation**.

Kalman-Yakubovich-Popov Lemma

Suppose (A, B) is completely controllable, and $\det(i\omega I - A) \neq 0$ for $\omega \in \mathbb{R}$.

Then the following are equivalent.

(1) The Hermitian matrix

$$\begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}$$

is positive semidefinite for all $\omega \in \mathbb{R} \cup \{\infty\}$.

(2) There exists a Hermitian matrix P such that

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} + \begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix}$$

is positive semidefinite.

If R is nonsingular, P satisfies the **Riccati inequality**

$$Q + A^*P + PA - (S + PB)R^{-1}(S + PB)^* \geq 0.$$

Important matrix equations in Linear Systems Theory

1) Sylvester equations:

$$AX - XB = \pm C, \qquad \dot{X} = AX - XB + C.$$

2) Lyapunov equations:

$$A^*P + PA = \pm Q, \qquad \dot{P} = A^*P + PA + Q.$$

3) nonsymmetric Riccati equations:

$$0 = DX - XA + C - XBX, \qquad \dot{X} = DX - XA + C - XBX.$$

4) symmetric Riccati equations:

$$0 = A^*P + PA + Q - PRP, \qquad -\dot{P} = A^*P + PA + Q - PRP.$$

Notation: $A^* = \bar{A}^\top$ is the conjugate transpose of A.

Structured Pseudospectra

Data:

Nominal matrix	:	$A \in \mathbb{C}^{n\times n}$
Perturbation class	:	struct $\subseteq \mathbb{C}^{n\times n}$ subspace (over $\mathbb{C}$ or $\mathbb{R}$)
$\|\cdot\|$	:	norm on struct
Perturbation level	:	$\rho \geq 0$.

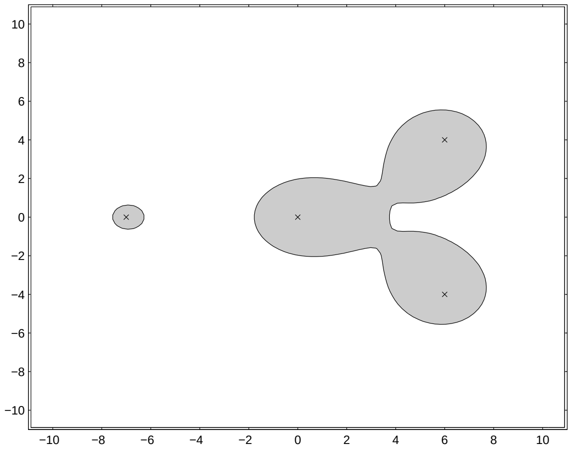
Definition:

$\sigma_{\text{struct}}(A,\rho)$:= Set of eigenvalues of all matrices of the form

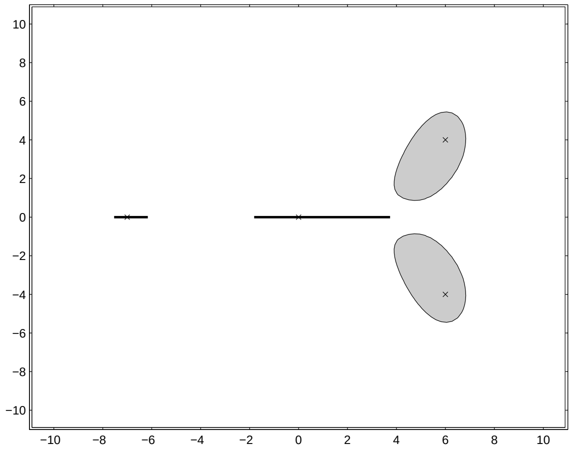
$A + \Delta, \qquad \Delta \in \text{struct}, \ \|\Delta\| \leq \rho.$

Example:

$\rho = 1.0$



struct $= \mathbb{C}^{n\times n}$



struct $= \mathbb{R}^{n\times n}$