Indian Institute of Technology Kharagpur
Course: MA51014/MA41004 Topology*
Spring Semester 2016
Assignment -02

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1. Prove that a topology τ defined on a nonempty set X is the discrete topology if and only if all the singleton subsets of X are open sets, that is, $\{x\} \in \tau$, for all $x \in X$.

2. Find three topologies τ_1, τ_2 and τ_3 ($\tau_i, i = 1 : 3$ is not the trivial or the discrete topology) on the set $X = \{a, b, c, d, e\}$ such that τ_1 is finer than τ_2, and τ_2 is finer than τ_3. Find one more topology τ_4 on X that is not comparable to any of τ_1, τ_2 and τ_3.

3. Let X be a set and $p \in X$. Show that the collection τ consists of all the subsets of X containing p, the null set and X, is a topology on X.

4. Determine which of the following collections of subsets of $\mathbb{R}$ are bases:
   (a) $\mathcal{B}_1 = \{(a,\, b) \cup \{b+1\} \subset \mathbb{R} \,|\, a < b\}$
   (b) $\mathcal{B}_2 = \{(n,\, n+2) \subset \mathbb{R} \,|\, n \in \mathbb{Z}\}$
   (c) $\mathcal{B}_3 = \{[a,\, b] \subset \mathbb{R} \,|\, a \leq b\}$

5. An open half plane in $\mathbb{R}^2$ is a subset of $\mathbb{R}^2$ of the form $\{(x,\, y) : ax + by < c\}$ for some $a, b, c \in \mathbb{R}$ with either a or b nonzero. Prove that these open half planes are open subsets of $\mathbb{R}^2$ in the standard topology on $\mathbb{R}^2$.

6. Can there be two sub-bases for the standard topology on $\mathbb{R}$?

7. Show that the lower limit topology on $\mathbb{R}$ is Hausdorff.

8. Determine $\text{Int}(A)$ and $\text{Cl}(A)$ in each of the following cases:
   (a) $A = \{a\}$ in $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$.
   (b) $A = (-1,\, 1) \cup \{2\}$ in the lower limit topology on $\mathbb{R}$.

9. Show that the set $S = \{\frac{a}{2^n} \,|\, a \in \mathbb{Z}, n \in \mathbb{Z}_+\}$ is dense in $\mathbb{R}$ with standard topology.

10. For each $n \in \mathbb{Z}_+$, let $B_n = \{n, n+1, n+2, \ldots\}$, and consider the collection $\mathcal{B} = \{B_n \,|\, n \in \mathbb{Z}_+\}$. Then
    (a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}_+$.
    (b) Show that the topology generated by $\mathcal{B}$ is not Hausdorff.
    (c) Show that the sequence $\{2, 4, 6, 8, \ldots\}$ converges to every point point in $\mathbb{Z}_+$.

11. Determine the set of limit points of $[0,\, 1]$ in $\mathbb{R}$ where $\mathbb{R}$ is with finite complement topology.

12. Determine the set of limit points of the set $S = \{(x,\, \sin(\frac{1}{x})) \in \mathbb{R}^2 \,|\, 0 < x \leq 1\}$ in $\mathbb{R}^2$ equipped with the standard topology.

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*Instructor: Bibhas Adhikari, bibhas@maths.iitkgp.ernet.in

13. Determine ∂A when

    (a) $A = \{(\frac{1}{n}, 0) \in \mathbb{R}^2 \,|\, n \in \mathbb{Z}_+\}$, $\mathbb{R}^2$ with standard topology.

    (b) $A = [0,\, 1] \subset \mathbb{R}$ with cofinite topology.

14. Let (X, τ_X) be a topological space and (Y, τ_Y) a subspace of X. Show that a set $C \subset Y$ is closed if and only if $C = D \cap Y$ for some closed set D in X.

15. Show that a subspace of a Hausdorff topological space is Hausdorff.

16. Prove the following:

    (a) If X and Y are Hausdorff spaces then so is the product space $X \times Y$.

    (b) If A is closed in (X, τ_X) and B is closed in (Y, τ_Y) then so is $A \times B$ in $(X \times Y, \tau_X \times \tau_Y)$.

17. Let X and Y be topological spaces and $A \subset X, B \subset Y$. Then

    (a) Provide an example demonstrating that $\partial(A \times B) = \partial A \times \partial B$ does not hold in general.

    (b) Derive and prove a relationship expressing $\partial(A \times B)$ in terms of ∂A, ∂B, A and B.