MA 20205 Probability and Statistics
Hints/Solutions to Assignment No. 6

1. $P(1 < X + Y < 2)$

$$= \int_0^1 \int_{1-y}^{2-y} e^{-(x+y)} dxdy + \int_1^2 \int_0^{2-y} e^{-(x+y)} dxdy)$$
$$= (e^{-1} - e^{-2}) + (e^{-1} - 2e^{-2}) = (2e^{-1} - 3e^{-2}).$$

$P(X < Y | X < 2Y) = \frac{P(X<Y)}{P(X<2Y)}$. Now

$$P(X < Y) = \int_0^\infty \int_x^\infty e^{-(x+y)} dydx = \frac{1}{2}$$
$$P(X < 2Y) = \int_0^\infty \int_{\frac{x}{2}}^\infty e^{-(x+y)} dydx = \frac{2}{3}$$

So the required probability is 0.75.

Clearly X and Y are independent. So

$P(0 < X < 1 | Y = 2) = P(0 < X < 1) = 1 - e^{-1}$.

$P(X + Y < m) = \frac{1}{2}$ is equivalent to $2(m + 1)e^{-m} - 1 = 0$. This is a nonlinear equation and can be solved numerically. Elementary numerical methods such as bisection gives $m \approx 1.68$.

2. The marginal densities of X and Y are

$f_X(x) = x + \frac{1}{2}$, $0 < x < 1$ and $f_Y(y) = y + \frac{1}{2}$, $0 < y < 1$.

Clearly X and Y are not independent.

$$E(X) = \frac{7}{12}, E(X^2) = \frac{5}{12}, V(X) = \frac{11}{144}$$
$$E(Y) = \frac{7}{12}, E(Y^2) = \frac{5}{12}, V(Y) = \frac{11}{144}$$
$$E(XY) = \frac{1}{3}, Cov(X, Y) = -\frac{1}{144}$$

$$Var(X+Y) = V(X) + V(Y) + 2Cov(X,Y) = \frac{5}{36}$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = -\frac{1}{11}$$

The conditional pdf of $X|Y = y$ is given by

$$f_{X|Y=y}(x|y) = \frac{2(x+y)}{2y+1} \quad ,0 < x < 1, 0 < y < 1\,.$$

$$E(X|Y=y) = \frac{2+3y}{3(2y+1)}, E(X^2|Y=y) = \frac{3+4y}{6(2y+1)},$$

$$V(X|Y=y) = \frac{6y^2+6y+1}{18(2y+1)^2}$$

3. The marginal densities of X and Y are respectively

$f_X(x) = \frac{1}{4}(3-x)\ ,0 < x < 2$ and $f_Y(y) = \frac{1}{4}(5-y)\ ,2 < y < 4$.

The conditional densities of $Y|X = x$ and $X|Y = y$ are respectively

$$f_{Y|X=x}(y|x) = \frac{(6-x-y)}{2(3-x)} \quad ,2 < y < 4, 0 < x < 2$$

$$f_{X|Y=y}(x|y) = \frac{(6-x-y)}{2(5-y)} \quad ,0 < x < 2, 2 < y < 4$$

We get $E(Y|X=x) = \frac{26-9x}{3(3-x)}\ ,E(Y^2|X=x) = \frac{78-28x}{3(3-x)}$.

$$E(X|Y=y) = \frac{14-3y}{3(5-y)}\ ,E(X^2|Y=y) = \frac{18-4y}{3(5-y)}\,.$$

Rest of the calculations can be done as before.

4. $(X,Y) \sim BVN(24, 28, 36, 49, 0.8)$. Then $X \sim N(24, 36)$. So

$P(X > 30) = \Phi(-1) = 0.1587$.

The conditional distribution of X given $Y = 35$ is $N(28.8, 12.96)$.

So $Var(X|Y = 35) = 12.96$.

$$P(X > 30|Y = 35) = P\left(Z > \frac{30-28.8}{\sqrt{12.96}}\right) = \Phi(-0.33) = 0.3707.$$

The conditional distribution of Y given $X = 22$ is $N(26.13, 17.64)$.

$E(Y|X=22)=26.13.$

5. Similar to Q. 4.
6. The marginal density of Y is $f_Y(y)=e^{-y}, y>0.$

   The conditional density of X given $Y=y$ is

   $f_{X|y=y}(x|y)=\frac{1}{y}\ e^{-\frac{x}{y}}\ , x>0.$

   $E(Y)=1, E(X)=EE(X|Y)=E(Y)=1. V(Y)=1.$

   $V(X)=VE(X|Y)+EV(X|Y)=V(Y)+E(Y^2)=1+2=3.$

   $E(XY)=E\big(Y\ E(X|Y)\big)=E(Y^2)=2.\ Cov(X,Y)=1.$

   $Corr(X,Y)=\frac{1}{\sqrt{3}}.$

7. Similar to Q. 4.
8. Similar to Q. 4.
9. In order that $f(x,y)$ is a valid density, $-1<\alpha<1.$

   The marginal densities of X and Y are

   $f_X(x)=1, 0<x<1$ and $f_Y(y)=1, 0<y<1.$

   $$E(X)=\frac{1}{2}\ ,\quad E(Y)=\frac{1}{2}\ , V(X)=\frac{1}{12}\ , V(Y)=\frac{1}{12}$$

   $$Cov(X,Y)=E\left(X-\frac{1}{2}\right)\left(Y-\frac{1}{2}\right)=-\frac{\alpha}{36}\ , Corr(X,Y)=-\frac{\alpha}{3}$$

   Clearly X and Y are independent if and only if $\alpha=0.$

10. a. $P(X\le 1, Y\ge 2)=p_{X,Y}(0,2)+p_{X,Y}(0,3)+p_{X,Y}(0,4)$

    $+p_{X,Y}(1,2)+p_{X,Y}(1,3)+p_{X,Y}(1,4)=0.356.$

    b. The conditional p.m.f.'s of X and Y are respectively

    $p_X(0)=0.210, p_X(1)=0.298, p_X(2)=0.277, p_X(3)=0.215\ .$

    $p_Y(1)=0.267, p_Y(2)=0.397, p_Y(3)=0.302, p_Y(4)=0.034\ .$

    $E(X)=1.497, E(Y)=2.103, E(X^2)=3.341, E(Y^2)=5.117$

    $Var(X)=1.099991, Var(Y)=0.694391,$

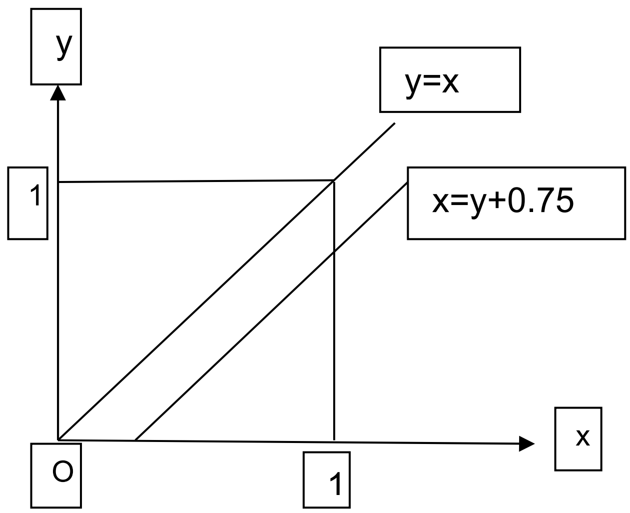
$E(XY) = 3.279, Cov(X,Y) = 0.130809, Corr(X,Y) \approx 0.1498.$

c. $P(Y \geq 2|X = 1) = \frac{p_{X,Y}(1,2)+p_{X,Y}(1,3)+p_{X,Y}(1,4)}{p_X(1)} \approx 0.6897.$

11. X →time to switch off the lights

Y → time to switch on the lights

(a) The desired region is the area between the lines $y = x$ and $x = y + 0.75$ inside the unite square as shown in the figure below:



The required probability $= P(Y < 0.5, X < Y + 0.25)$

$$= \int_0^{0.75} \int_y^{y+0.25} 8xy\, dx\, dy + \int_{0.75}^1 \int_y^1 8xy\, dx\, dy = \frac{139}{256} \approx 0.543.$$

(b) The marginal density of Y is

$$f_y(y) = 4y(1 - y^2), 0 < y < 1.$$

The conditional density of X given $Y = y$ is

$$f_{X|Y=y}(x|y) = \frac{2x}{1-y^2}, \quad y < x < 1 \text{ for } 0 < y < 1.$$

So the conditional density of X given $Y = \frac{1}{6}$ is

$$g(x) = \frac{72\,x}{35}, \frac{1}{6} < x < 1.$$

The required probability

$= P\left(X < \frac{3}{4} | Y = \frac{1}{6}\right) = \int_{\frac{1}{6}}^{\frac{3}{4}} g(x) dx = \frac{17}{35}$.

(c) $E\left(X | Y = \frac{1}{6}\right) = \frac{43}{63}$

(d) The marginal density of X is

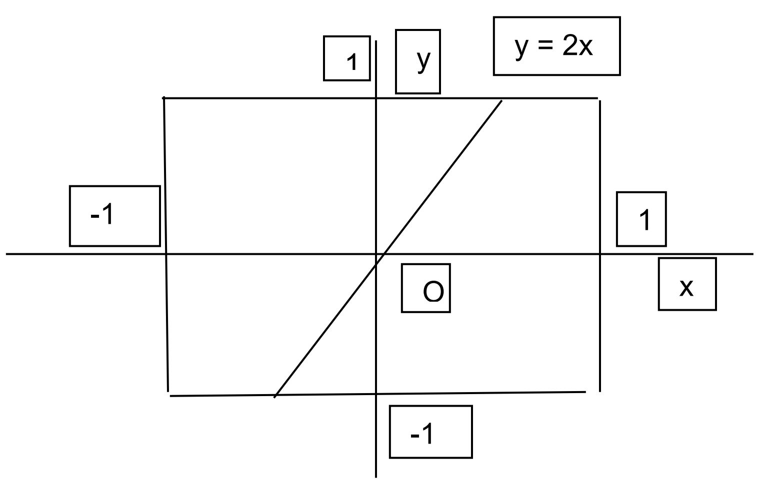
$f_X(x) = 4x^3, \quad 0 < x < 1$.

$E(X) = \frac{4}{5}, E(Y) = \frac{8}{15}, E(X^2) = \frac{2}{3}, E(Y^2) = \frac{1}{3}, E(XY) = \frac{4}{9}$.

$$Var(X) = \frac{6}{25}, Var(Y) = \frac{11}{225}, Cov(X,Y) = \frac{4}{225}$$

$$Corr(X,Y) = \frac{4}{3\sqrt{66}} = 0.1641 .$$

12. The desired area is the region to the left of the line $y = 2x$ in the square below.



$$P(2X < Y) = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{\frac{y}{2}} (1 + xy)\, dx\, dy = \frac{1}{2}$$

The desired area is the region between lines $x + y = -1$ and $x + y = 1$ in the same square.

As the density is symmetric in X and Y

$$P(|X+Y|<1) = 2\,.\frac{1}{4}\int_0^1\int_{-1}^{1-y}(1+xy)\,dx\,dy = \frac{31}{48}.$$

The marginal distributions of X and Y are $U(-1,1)$.

$$E(X) = E(Y) = 0, Var(X) = Var(Y) = \frac{1}{3}, E(XY) = \frac{1}{9}$$

$$Cov(X,Y) = \frac{1}{9}, Corr(X,Y) = \frac{1}{3}$$

13. $c = \frac{1}{6}$. The marginal densities of X and Y are

$f_X(x) = 6x(1-x),\ 0 < x < 1$ and

$f_Y(y) = 6(\sqrt{y} - y), 0 < y < 1.$

$$P\left(\frac{1}{3} < X < \frac{2}{3}\right) = \frac{13}{27}, P\left(\frac{1}{4} < Y < \frac{3}{4}\right) = \frac{3\sqrt{3}-4}{2} \approx 0.598$$

The conditional densities of $X|Y = y$ and $Y|X = x$ are respectively

$$f_{X|Y=y}(x|y) = \frac{1}{\sqrt{y}-y} \quad , y < x < \sqrt{y} \quad , 0 < y < 1$$

$$f_{Y|X=x}(y|x) = \frac{1}{x - x^2} \quad , x^2 < y < x, \qquad 0 < x < 1$$

The conditional density of $X|Y = \frac{1}{4}$ is

$g(x) = 4\ , \frac{1}{4} < x < \frac{1}{2}$. So $P\left(\frac{5}{16} < X < \frac{7}{16} | Y = \frac{1}{4}\right) = \frac{1}{2}$

The conditional density of $Y|X = \frac{1}{2}$ is

$h(y) = 4, \frac{1}{4} < y < \frac{1}{2}$. So $P\left(\frac{3}{8} < Y < \frac{5}{8} \mid X = \frac{1}{2}\right) = \frac{1}{2}$.
The region $|X - Y| < 0.5$ contains the region of the density $x^2 < y < x, 0 < x < 1$. So $P(|X - Y| < 0.5) = 1$.

14. The marginal densities of X_1 and X_2 are given by

$$f_{X_1}(x_1) = \frac{1}{2}(1 + x_1)e^{-x_1} \ , x_1 > 0 \text{ and}$$

$$f_{X_2}(x_2) = \frac{1}{2}(1 + x_2)e^{-x_2} \ , x_2 > 0$$

The conditional densities of $X_1|X_2 = 2$ and $X_2 \, |X_1 = 4$ are

$$g(x_1) = \frac{1}{3}(x_1 + 2)e^{-x_1} \ , x_1 > 0 \text{ and}$$

$$h(x_2) = \frac{1}{5}(x_2 + 4)\, e^{-x^2} \ , x_2 > 0.$$

$E(X_2|X_1 = 4) = \frac{6}{5}, Var(X_1|X_2 = 2) = \frac{14}{9}$

$E(X_1) = E(X_2) = \frac{3}{2}, V(X_1) = V(X_2) = \frac{7}{4}, E(X_1X_2) = 2$

$Cov(X_1, X_2) = -\frac{1}{4}, Corr(X_1, X_2) = -\frac{1}{7}$.

15. Here $\mu_1 = -1, \mu_2 = 1, \sigma_1^2 = 4, \sigma_2^2 = 9, \rho = -0.5$. Then $X \sim N(-1, 4), Y \sim N(1, 9), X|Y = y$ has $N\left(\frac{-2-y}{3}, 3\right)$ and $Y|X = x$ has $N\left(\frac{1-3x}{4}, \frac{27}{4}\right)$.

$$E\{(X + 1)^2(Y - 1)^2\} = E^Y[(Y - 1)^2 E\{(X + 1)^2|Y\}]$$

$$= E\left[(Y - 1)^2\left\{\frac{(1-Y)^2}{9} + 9\right\}\right] = \frac{1}{9}\, E(Y - 1)^4 + 9E(Y - 1)^2$$

$$= \frac{1}{9}\, .3.9^2 + 9.9 = 108.$$

$P(X < 1) = \Phi(1) = 0.8413, P(Y > 4) = \Phi(-1) = 0.1587.$

Now $X|Y=\frac{3}{2}$ follows $N\left(-\frac{7}{6},3\right)$.

So $P\left(-\frac{3}{2}<X<-\frac{1}{2}|Y=\frac{3}{2}\right)=\Phi\left(\frac{2}{3\sqrt{3}}\right)-\Phi\left(-\frac{1}{3\sqrt{3}}\right)$

$=\Phi(0.39)-\Phi(-0.19)=0.6517-0.4247=0.227.$

Also $Y|X=-\frac{1}{2}$ follows $N\left(\frac{5}{8},\frac{27}{4}\right)$.

So $P\left(\frac{1}{2}<Y<\frac{3}{2}\middle|X=-\frac{1}{2}\right)=\Phi\left(\frac{7}{12\sqrt{3}}\right)-\Phi\left(-\frac{1}{12\sqrt{3}}\right)$

$=\Phi(0.34)-\Phi(0.05)=0.6331-0.5199=0.1132.$

16. Here $(X,Y)\sim BVN\left(1,1,1,1,\frac{1}{3}\right)$. So $2X+3Y\sim N(5,17)$.
Remaining parts can be answered as before.

17. Q. 17 can be solved as earlier questions 12, 13 etc..