MA 20205 Probability and Statistics
Hints/Solutions to Assignment No. 4

1. $R_X(t) = P(X > t) = Exp\{-\int Z_X(t)dt\} = Exp\{-(0.5\,t + t^2)\}$.

$P(X > 2|X > 1) = \frac{P(X>2)}{P(X>1)} = \frac{R_X(2)}{R_X(1)} = \frac{e^{-5}}{e^{-1.5}} = e^{-3.5} \cong 0.0302$

2. Let X denote the marks and follow normal distribution with mean μ and standard deviation .
Given

$$P(X \le 45) = 0.1 \Rightarrow P\left(\frac{X-\mu}{\sigma} \le \frac{45-\mu}{\sigma}\right) = 0.1$$
$$\Rightarrow \Phi\left(\frac{45-\mu}{\sigma}\right) = 0.1 \Rightarrow \frac{45-\mu}{\sigma} = -1.28 \quad \cdots (1)$$

$$P(X > 75) = 0.05 \Rightarrow P\left(\frac{X-\mu}{\sigma} > \frac{75-\mu}{\sigma}\right) = 0.05$$
$$\Rightarrow \Phi\left(\frac{75-\mu}{\sigma}\right) = 0.95 \Rightarrow \frac{75-\mu}{\sigma} = 1.65 \cdots (2)$$

Solving (1) and (2), we get $\mu = 58.11, \sigma = 10.24$.
Percentage of students getting Ist Class
$= P(60 < X \le 75) = P(X \le 75) - P(X \le 60) = 0.95 - \Phi\left(\frac{60-58.11}{10.24}\right) = 0.95 - \Phi(0.18)$
$= 0.95 - 0.5714 = 0.3786$. So the percentage of students getting the Ist Class is 37.86%.

Percentage of students getting IInd Class
$= P(45 < X \le 60) = P(X \le 60) - P(X \le 45) = 0.5714 - 0.1 = 0.4714$.

So 47.14%.

3. Let X denote the diameter (in cm) of a ball bearing. Then $X \sim N(3, 0.005^2)$.
$\mathrm{P}(\text{ball bearing is scrapped}) = 1 - \mathrm{P}(2.99 < \mathrm{X} < 3.01)$
$= 2\Phi(-2) = 0.0455$.
So approximately 4.55% of ball bearings will be scrapped.

4. $P(0.895 < X < 0.905) = P(-1.66 < Z < 1.66) = 2\,\Phi(1.66) - 1 = 0.903$.

So the percentage of defectives $100 \times 0.097 = 9.7\%$.

When $X \sim N(0.9, \sigma^2)$, then $P(0.895 < X < 0.905) \ge 0.99$ is equivalent to

$\Phi\left(\frac{0.005}{\sigma}\right) \geq 0.995$ or $\sigma \leq 0.00194$.

5. Let X denote the height (in cm.) that university high jumper jumps.
   Then $\sim N(200, 100)$.
   Let c be such that $P(X > c) = 0.95 \Rightarrow \frac{200-c}{10} = 1.645 \Rightarrow c = 183.55\ cm.$
   Further let d be such that $P(X > d) = 0.1 \Rightarrow \frac{200-d}{10} = -1.28 \Rightarrow d = 212.80\ cm.$

6. Let X denote the marks. Then $X \sim N(74, 62.41)$.
   Using tables of normal distribution, we get
   Ans. (a) 64 (b) 86 (c) 77

7. E(Profit) $= C_0 P(6 \leq X \leq 8) - C_1 P(X < 6) - C_2 P(X > 8)$
   $= C_0\{\Phi(8-\mu) - \Phi(6-\mu)\} - C_1 \Phi(6-\mu) - C_2 \Phi(\mu - 8)$

   Using derivatives and simplifying we obtain the maximizing choice of μ as

   $$\mu^* = 7 + \frac{1}{2}\ln\left(\frac{C_1 + C_0}{C_2 + C_0}\right)$$

8. Let X denote the IQ levels of candidates. Then $X \sim N(90, 25)$.
   $P(85 < X < 95) = \Phi(1) - \Phi(-1) = 0.8413 - 0.1587 = 0.6826 = p$, say
   Let Y denote the number of candidates with IQ levels between 85 and 95.
   Then $Y \sim Bin(4, p)$.
   $P(Y \geq 2) = 1 - P(Y = 0) - P(Y = 1)$
   $= 1 - (0.3174)^4 - 4(0.3174)^3(0.6826) \approx 0.9025$

9. $P(X > 2) = \frac{1}{4} = p$, say. Let Y denote the number of machines which are working for more than 2 years. Then $Y \sim Bin(200, 0.25)$. As n is large, $\frac{Y-50}{6.1237} \approx Z \sim N(0,1)$.
   $P(X \geq 60) \approx P\left(Z \geq \frac{59.5-50}{6.1237}\right) = \Phi(-1.55) = 0.0606.$

10. Let X denote the number of students who cannot solve the question. Then $X \sim Bin(200, 0.5)$. Since As n is large, $\frac{X-100}{7.071} \approx Z \sim N(0,1)$.
    $P(X \geq 110) \approx P\left(Z \geq \frac{109.5-100}{7.071}\right) = \Phi(-1.34) = 0.901.$

11. Let X be the number of trains arriving/ departing between 2:00 p.m. to 3:00 p.m. Then $X \sim \mathcal{P}(20)$. As λ is large, $\frac{X-20}{4.4721} \approx Z \sim N(0,1)$.

$P(17 \leq X \leq 25) \approx P\left(\frac{16.5-20}{4.4721} \leq Z \leq \frac{25.5-20}{4.4721}\right) = \Phi(1.23) - \Phi(-0.78)$
$= 0.89065 - 0.2177 = 0.67295.$

12. $\ln Y \sim N(0.8, 0.01)$.
$P(Y > 2.7) = P(\ln Y > 0.9933) = P(Z > 1.93)$
$= \Phi(-1.93) = 0.0268.$
Let c be such that $P(0.8 - c < \ln Y < 0.8 + c) = 0.95$. This is equivalent to
$P\left(-\frac{c}{0.1} < Z < \frac{c}{0.1}\right) = 0.95,$
or $\Phi\left(\frac{c}{0.1}\right) = 0.975 \;\Rightarrow \frac{c}{0.1} = 1.96 \Rightarrow c = 0.196.$
So the desired interval is $(1.8294, 2.7074)$
13. Let $X \sim Beta(\alpha, \beta)$. Then $E(X) = \frac{\alpha}{\alpha+\beta} = \frac{1}{3}$ and $(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{18}$.
$\Rightarrow E(X^2) = \frac{\alpha(\alpha+\beta)}{(\alpha+\beta)(\alpha+\beta+1)} = \frac{1}{2}$. This gives $\alpha = 2, \beta = 1$.
So the pdf of X is $f_X(x) = 2x \;,\; 0 < x < 1.$
$P(0.2 < X < 0.5) = 0.21.$

14. $E\left(\frac{1}{X+1}\right) = \frac{e^{-\lambda}}{\lambda(1-e^{-\lambda})} \sum_{x=1}^{\infty} \frac{\lambda^{x+1}}{(x+1)!} = \frac{e^{-\lambda}(e^{\lambda}-\lambda-1)}{\lambda(1-e^{-\lambda})} = \frac{(e^{\lambda}-\lambda-1)}{\lambda(e^{\lambda}-1)}$