MA 20205 Probability and Statistics
Hints/Solutions to Assignment No. 3

1.

$$P(\text{Ruby wins}) = \frac{\binom{4}{2}\binom{3}{1} + \binom{4}{3}}{\binom{7}{3}} = \frac{22}{35} = 0.6286.$$

2. Suppose n missiles are fired and X is the number of successful hits.
   Then $X \sim \text{Bin}(n, 0.75)$.
   We want n such that $P(X \geq 3) \geq 0.95$, or $P(X = 0) + P(X = 1) + P(X = 2) \leq 0.05$. This is equivalent to $10(9n^2 - 3n + 2) \leq 4^n$.
   The smallest value of n for which this is satisfied is $n = 6$.

3. Let X be the number of defectives in a sample of three items.
   Then $X \sim Bin(3, 0.1)$
   The required probability $= P(X \leq 1) = (0.9)^3 + \binom{3}{1}(0.9)^2(0.1) = 0.972.$

4. $np = 8, npq = 4$. This gives $n = 16, p = 0.5$.
   So $P(X \geq 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$

$$= 1 - \left(\frac{1}{2}\right)^{16} - 16\left(\frac{1}{2}\right)^{16} - 120\left(\frac{1}{2}\right)^{16} = \frac{65399}{65536} = 0.9979.$$

5. Let P_n denote the probability of an n-component system to operate effectively. Then

$$P_3 = 1 - (1-p)^3 \; and \; P_6 = 1 - (1-p)^6 - \binom{6}{1} p(1-p)^5 .$$

   Now $P_6 - P_3 \geq 0$, if $\frac{9 - \sqrt{21}}{10} \leq p \leq 1$.

6. This is mgf of a geometric distribution with $p = \frac{2}{7}$. Using memoryless property of the geometric distribution, the required probability $= P(X > 2) = \left(\frac{5}{7}\right)^2 = \frac{25}{49}$.

7.

$$P(\text{returning a packet}) = P(X > 1) = 1 - P(X = 0) - P(X = 1)$$
$$= 1 - (0.99)^{10} - 10(0.99)^9(0.01), \text{ say.}$$

Let Y denote the number of packets returned. Then $Y \sim \text{Bin}(3, p)$.
$P(Y = 0) + P(Y = 1) = 0.9999455.$

8. X has NB(3, 08) distribution.

$$P(X=k)=\binom{k-1}{2}(0.2)^{k-3}(0.8)^3,\quad k=3,4,\ldots$$

The required probability = $P(X\geq 5)=1-P(X=3)-P(X=4)$

$=1-(0.8)^3-3(0.2)(0.8)^3=0.1808.$

9. Required Probability $=\dfrac{\binom{5}{1}\binom{15}{3}}{\binom{20}{4}}=\dfrac{455}{969}=0.4696.$

10.

$$\mathrm{P(Y=2000j)}=\frac{e^{-2}(2)^j}{j!},\qquad j=0,1,\cdots,9,$$

$$=\sum_{j=10}^{\infty}\frac{e^{-2}(2)^j}{j!},\quad j=10.$$

11. Let A be the event that person gets a cold and B denote the event that the drug is beneficial to him. Let X be the number of times an individual contracts the cold in a year. Then $X\,|\,B\sim P(2)$, and $X\,|\,B^C\sim P(3)$.

$P(A^C\,|\,B)=P(X=0\,|\,B)=e^{-2}.\ \ P(A^C\,|\,B^C)=P(X=0\,|\,B^C)=e^{-3}.$

Using Bayes Theorem $P(B\,|\,A^C)=0.89.$

12. Let X be the number of errors. Then X ~ P(300). Let Y be the number of errors in 2% of the pages. Then Y ~ P(6). The required probability is $P(Y\leq 4)\cong 0.2851$.

13. Use Geometric probability to evaluate. The required probability = 5/9.

14. This is MGF of discrete uniform distribution on points 1, 2, …, 10. The variance is

$\dfrac{N^2-1}{12}=\dfrac{99}{12}=8.25.$

15. $X\sim U\left(\dfrac{3C}{4},2C\right)$. Let b be the bid by the contractor and let the profit be $P(X)$. Then

$$P(X)=\begin{cases}0, & \text{if } X<b\\ b-C & \text{if } X\geq b\end{cases}.\quad EP(X)=\frac{4}{5C}(b-C)(2C-b)=g(b),\text{ say.}$$

g(b) is maximized at $b=\dfrac{3C}{2}$.

16. Let X denote the life of a bulb. Then $P(X > 100) = e^{-2}$. Let Y denote the number of bulbs working after 100 hours. Then $Y \sim Bin(10, e^{-2})$. So $P(Y \geq 2) \approx 0.4008$.

17. $P(X > 5) = \frac{1}{4}P(X > 5 \mid A) + \frac{3}{4}P(X > 5 \mid B) = 0.25\, e^{-1} + 0.75\, e^{-2.5} \approx .1535.$

18. Let X denote the life of a motherboard. Then $f_X(x) = \frac{1}{2}e^{-x/2}, \quad x > 0$. Let $P(X)$ be the profit.

Then

$$P(x) = \begin{cases} 3000, & x < 1/2 \\ 5000, & x \geq 1/2 \end{cases}.$$

$$\begin{aligned} EP(X) &= 3000P(X < 1/2) + 5000P(X \geq 1/2) \\ &= 3000(1 - e^{-1/4}) + 5000\, e^{-1/4} \\ &= 3000 + 2000\, e^{-1/4} = 4557.60 \end{aligned}$$

19. $P(\text{system fails before time t}) = 1 - \exp\{-t\sum_{i=1}^{n}\lambda_i\}$.

$P(\text{only component j fails before time t} \mid \text{system fails before time t})$

$$= \frac{(1\text{-}\exp\{\text{-}\lambda_j t\})\exp\{-t\sum_{\substack{i=1 \\ i \neq j}}^{n}\lambda_i\}}{1 - \exp\{-t\sum_{i=1}^{n}\lambda_i\}}.$$

20. Let X denote the life of an AC. Then $f_X(x) = \frac{1}{100}e^{-x/100}, \quad x > 0$.

$P(X > 100) = e^{-1}$. Let Y denote the number of AC s working after 100 hours. Then $Y \sim Bin(5, e^{-1})$. $P(Y \geq 2) \approx 0.6054$.

21. The required probability = $P(X < 15) = 1 - e^{-15/10} \approx 0.7769$.

22. Let X denote the lead time for orders of diodes. Then $E(X) = \frac{r}{\lambda} = 20, V(X) = \frac{r}{\lambda^2} = 100$.

Then $r = 4, \lambda = 1/5$. So $f_X(x) = \frac{1}{6.5^4}e^{-x/5}x^3, \quad x > 0$.

The required probability = $P(X < 15) = 1 - 13e^{-3} = 0.3528$.

23. Let X denote the life of an electronic equipment. For manufacturer ‘A’,

$E(X) = \frac{r}{\lambda} = 4, V(X) = \frac{r}{\lambda^2} = 8$. Then $r = 2, \lambda = 1/2$. So $f_{X|A}(x) = \frac{x}{4}e^{-x/2}, \quad x > 0$.

For manufacturer 'B', $E(X) = \frac{r}{\lambda} = 2, V(X) = \frac{r}{\lambda^2} = 4$. Then $r = 1, \lambda = 1/2$. So

$f_{X|B}(x) = \frac{1}{2} e^{-x/2}, \quad x > 0$. Given $P(A) = 0.75, \quad P(B) = 0.25$.

$$P(X > 12) = 0.75\, P(X > 12 \mid A) + 0.25\, P(X > 12 \mid B) = 0.25 \times 7e^{-6} + 0.25 \times e^{-6}$$
$$= 2\, e^{-6} \approx 0.005.$$

24. Let X denote the printing time. For Printer I, $f_{X|I}(x) = \frac{1}{3} e^{-x/3}, \quad x > 0$. For Printer II,

$E(X) = \frac{r}{\lambda} = 2, V(X) = \frac{r}{\lambda^2} = 2$. Then $r = 2, \lambda = 1$. So $f_{X|II}(x) = x\, e^{-x}, \quad x > 0$.

For Printer III, $f_{X|III}(x) = \frac{1}{4}, \quad 0 < x < 4$. Given $P(I) = 0.3, \quad P(II) = 0.3, \; P(III) = 0.4$.

$$P(X < 1) = 0.3\, P(X < 1 \mid I) + 0.3\, P(X < 1 \mid II) + 0.4\, P(X < 1 \mid III)$$
$$= 0.3\,(1 - e^{-1/3}) + 0.3(1 - 2e^{-1}) + 0.4(0.25) \approx 0.2643.$$

25. $P(X \le 100 \mid X \ge 90) = 0.15$ gives $\alpha = 0.0000855$.

$P\left(X > 80\right) = \exp\{-0.0000855 \times 80^2\} = 0.5786$.