MA 20205 Probability and Statistics
Hints/Solutions to Assignment No. 2

1. $c = \frac{2}{3}$.

$$F(x) = \begin{cases} 0, & x \le 0 \\ \frac{x^3}{3}, & 0 < x \le 1. \\ 1 - \frac{2}{3x}, & x > 1 \end{cases}$$

$E(X)$ and $Var(X)$ do not exist.

$Med(X) = \frac{4}{3}$, $P(0.5 < X < 2) = \frac{5}{8}$, $P(X > 3) = \frac{2}{9}$.

2. $P\left(\frac{1}{2} < X < \frac{5}{2}\right) = F\left(\frac{5}{2}-\right) - F\left(\frac{1}{2}\right) = \frac{11}{12} - \frac{1}{8} = \frac{19}{24}$.

$P(1 < X < 3) = F(3-) - F(1) = \frac{11}{12} - \frac{1}{2} = \frac{5}{12}$.

The random variable X is continuous in the intervals (0, 1) and (1, 2) with the uniform density $f(x) = \frac{1}{4}$, and discrete at points 1, 2 and 3 with probabilities 1/4, 1/6 and 1/12 respectively. So

$$E(X) = \int_0^1 \frac{x}{4}\,dx + \int_1^2 \frac{x}{4}\,dx + 1.\frac{1}{4} + 2.\frac{1}{6} + 3.\frac{1}{12} = \frac{4}{3}.$$

$$E(X^2) = \int_0^1 \frac{x^2}{4}dx + \int_1^2 \frac{x^2}{4}dx + 1^2.\frac{1}{4} + 2^2.\frac{1}{6} + 3^2.\frac{1}{12} = \frac{7}{3}.$$

$V(X) = \frac{5}{9}$, $Median\,(X) = 1$.

3. Let S denote a survival and D denote a death of a guinea pig during the trial, then the sample space for this experiment can be described by

$\Omega = \{SS, SDS, SDD, DSS, DSD, DDSS, DDSD, DDDS, DDDD\}$

and the probabilities associated with these sample points are given by 4/9, 4/27, 2/27, 4/27, 2/27, 4/81, 2/81, 2/81, 1/81 respectively.

Let X = the number of survivors, Y = number of deaths.

Then the pmf of X is

$P(X = 2) = P(\{SS, SDS, DSS, DDSS\}) = 64/81$,

$P(X = 1) = P(\{SDD, DSD, DDSD, DDDS\}) = 16/81$,

$P(X = 0) = P(\{DD\}) = 1/81$.

The pmf of Y is

$P(Y = 0) = P(\{SS\}) = 4/9$, $P(Y = 1) = P(\{SDS, DSS\}) = 8/27$,

$P(Y = 2) = P(\{SDD, DSD, DDSS\}) = 16/81$,

$P(Y = 3) = P(\{DDSD, DDDS\}) = 4/81$, $P(Y = 4) = P(\{DDDD\}) = 1/81$.

4. Let X = the number of second generation particles,
   Let Y = the number of third generation particles,
   Then $X \to 1, 2, 3$; $Y \to 1, 2, \ldots, 9$.

   $P(Y = 1) = 1/9$, $P(Y = 2) = 4/27$, $P(Y = 3) = 16/81$,

   $P(Y = 4) = 4/27 = P(Y = 5)$,

   $P(Y = 6) = 10/81$, $P(Y = 7) = 2/27$,

   $P(Y = 8) = 1/27$, $P(Y = 9) = 1/81$.

5. Let X denote the scores on IQ test.

   $$P(X < 52 \text{ or } X > 148) = P(|X - 100| > 48) \le \frac{V(X)}{(48)^2} = \frac{1}{9}.$$

6. The cdf is given by

   $$F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x^2}{4}, & 0 \le x \le 1 \\ \dfrac{2x-1}{4}, & 1 < x \le 2 \\ \dfrac{6x - x^2 - 5}{4}, & 2 < x \le 3 \\ 1, & x \ge 3 \end{cases}.$$

   $E(X) = 3/2$, $Median\ (X) = 3/2$, $Var(X) = 5/12$.

7. Note that $\int_{-\infty}^{\infty} f(x)\, dx = 1$ for all $k \in \Re$. To have $f(x) \ge 0 \quad \forall x \in \Re$, we must have $0 \le k \le 1$. The cdf is given by

$$F(x)=\begin{cases} 0, & x<0 \\ \dfrac{kx}{4}, & 0\le x\le 1 \\ \dfrac{k+x^2-1}{4}, & 1<x\le 2 \\ \dfrac{(3k+1)+(1-k)x}{4}, & 2<x<3 \\ 1, & x\ge 3 \end{cases}$$

In order to determine the median we note that the maximum value of $F(x)$ for $0\le x\le 1$ is $\frac{k}{4}$ and will be less than $\frac{1}{2}$. In the range $1<x\le 2$, it will be between $\frac{k}{4}$ and $\frac{k+3}{4}$ and so median will lie in this range. Equating $\frac{k+M^2-1}{4}$ to $\frac{1}{2}$ we get $M=\sqrt{3-k}$ and so $\sqrt{2}\le M\le\sqrt{3}$.

8. Let X be the number of tests required. Then X is either 1 or 11.

$P(X=1) = P(\text{ none has the disease}) = (0.99)^{10}$,

$P(X=11) = P(\text{ at least one has disease}) = 1-(0.99)^{10}$,

$E(X) = 11-10(0.99)^{10}$.

9. $P(X=i)=\frac{n_i}{m},\quad i=1,....,r,\qquad E(X)=\sum_{i=1}^{r}\frac{in_i}{m}$.

$$P(Y=i)=\frac{in_i}{\sum_{i=1}^{r} in_i},\ i=1,...r,\quad E(Y)=\frac{\sum_{i=1}^{r} i^2 n_i}{\sum_{i=1}^{r} in_i}.$$

The proof of $E(Y)\ \ge\ E(X)$ follows by noticing $E(U^2)\ge\{E(U)\}^2$.

10. $0\le p_X(i)\le 1,\quad i=1,\cdots,4$ yields $-\frac{1}{3}\le d\le\frac{1}{4}$.

$$E(X)=\frac{10-9d}{4},\ E(X^2)=\frac{30-47d}{4},\ V(X)=\frac{20-8d-81d^2}{16}.$$

$V(X)$ is minimized at $d=\frac{1}{4}$.

11. $\sum_{x=0}^{\infty} p_X(x)=k,$ so k = 1.

$$\begin{aligned} F(x) &= 0, \quad if \quad x<0, \\ &= 1/2, \quad if \quad 0\le x<1, \\ &= 2/3, \quad if \quad 1\le x<2, \\ &\ \vdots \\ &= n/(n+1), \quad if \quad n-1\le x<n, \\ &\ \vdots \end{aligned}$$

$E(X)$ does not exist. Any M between 0 and 1 is a median.

12. $P(40 < X < 60) = P(|X - \mu| < 10) \geq 1 - \frac{\sigma^2}{100} = 0.75.$

13. $X \sim Bin(4, 0.4)$. So $E(X) = 1.6, V(X) = \frac{24}{25}, P(X \leq 1) = \frac{297}{625}.$

As $p < 0.5$, the distribution is positively skewed.

14. In order to have two consecutive heads ending on n^{th} trial we must have sequence of the type ....$HTHT\underset{n-1}{H}\underset{n}{H}$. The probability of this is $\left(\frac{1}{2}\right)^n$.

Similarly in order to have two consecutive tails ending on n^{th} trial we must have sequence of the type ....$THTH\underset{n-1}{T}\underset{n}{T}$. The probability of this is also $\left(\frac{1}{2}\right)^n$.

So $P(X = n) = \frac{1}{2^{n-1}},\ n = 2, 3, 4, \ldots$

P(even number of tosses is required to end the experiment)

$$= \sum_{m=1}^{\infty} P(X = 2m) = \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{2m-1} = \left(\frac{2}{3}\right).$$

$$P(X = 6 \mid X = \text{even}) = \frac{P(X = 6)}{P(X = \text{even})} = \frac{3}{16}.$$

15. Let $U =$ upper face on dice of player A and $V =$ upper face on dice of player B.

Then $P(U = i) = \frac{i}{21},\ i = 1, 2, \ldots, 6$ and $P(V = j) = \frac{j^2}{91},\ j = 1, 2, \ldots, 6.$

$X = |U - V|$. So X can take values $0, 1, \ldots, 5$.

$$P(X = 0) = \sum_{i=1}^{6} P(U = i, V = i) = \frac{441}{1911}.$$

$$P(X = 1) = \sum_{i=1}^{5} P(U = i, V = i+1) + \sum_{i=1}^{5} P(U = i+1, V = i) = \frac{630}{1911}.$$

$$P(X = 2) = \sum_{i=1}^{4} P(U = i, V = i+2) + \sum_{i=1}^{4} P(U = i+2, V = i) = \frac{420}{1911}.$$

$$P(X = 3) = \sum_{i=1}^{3} P(U = i, V = i+3) + \sum_{i=1}^{3} P(U = i+3, V = i) = \frac{252}{1911}.$$

$$P(X = 4) = \sum_{i=1}^{2} P(U = i, V = i+4) + \sum_{i=1}^{2} P(U = i+4, V = i) = \frac{126}{1911}.$$

$$P(X = 5) = P(U = 1, V = 6) + P(U = 6, V = 1) = \frac{42}{1911}.$$