4-21-2008

Riccati Equations in Optimal Control Theory

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by

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Under the Direction of Michael Stewart

ABSTRACT

It is often desired to have control over a process or a physical system, to cause it to behave optimally. Optimal control theory deals with analyzing and finding solutions for optimal control for a system that can be represented by a set of differential equations. This thesis examines such a system in the form of a set of matrix differential equations known as a *continuous linear time-invariant system*. Conditions on the system, such as linearity, allow one to find an explicit closed form finite solution that can be more efficiently computed compared to other known types of solutions. This is done by optimizing a quadratic cost function. The optimization leads to solving a Riccati equation. Conditions are discussed for which solutions are possible. In particular, we will obtain a solution for a stable and controllable system. Numerical examples are given for a simple system with 2×2 matrix coefficients.

INDEX WORDS: Optimal Control, Matrix Equations, Riccati Equations.

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A Thesis Submitted in Partial Fulfillment of Requirements for the Degree of

Master of Science

in the College of Arts and Sciences

Georgia State University

2008

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Electronic Version Approved:

Office of Graduate Studies
College of Arts and Sciences
Georgia State University
May 2008

Acknowledgements

I would like to acknowledge and thank all those who have helped and/or supported me during the several years of working towards my Masters degree, completing my research and thesis, and successfully graduating. Especially my teacher and thesis advisor Dr. Michael Stewart, and my loving and understanding wife Karen. Most of all, I wish to give thanks and acknowledge the gifts and blessing that God has given me, that enabled me to do all this. Without God, I would be nothing. Through Him, I can do anything.

Contents

Chapter 1

Introduction

Many processes (electrical, mechanical, chemical, etc.) can be modeled by a linear time invariant system, which can be represented by the following system of matrix differential equations:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{1.1}$$

with initial condition $\mathbf{x}(0) = \mathbf{x}_0$,

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t). \tag{1.2}$$

Here A is $n \times n$, B is $n \times m$, C is $p \times n$, and D is $p \times m$. This is known as a *continuous linear time-invariant system*. Equation (1.1) is called a *state equation*. Equation (1.2) is called an *output equation*. The vector $\mathbf{x}(t)$ is the state of the system at time t. The vector $\dot{\mathbf{x}}(t)$ is the time derivative of the state of the system at time t. Vector $\mathbf{u}(t)$ is an independent input to the system. The vector $\mathbf{y}(t)$ is the output of the system. The matrices A, B, C, and D are matrices containing parameters of the overall system.

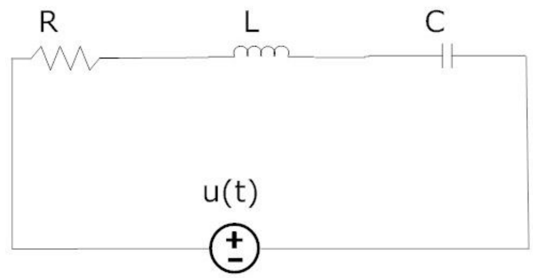

Figure 1.1: An RLC Circuit

The system given above has the properties of linearity and time-invariance. Linearity means that if input $\mathbf{u}_1(t)$ results in output $\mathbf{y}_1(t)$ and $\mathbf{u}_2(t)$ results in $\mathbf{y}_2(t)$, then

$$\alpha\mathbf{u}_1(t) + \beta\mathbf{u}_2(t) \to \alpha\mathbf{y}_1(t) + \beta\mathbf{y}_2(t).$$

Time-invariance means that if input $\mathbf{u}(t)$ results in output $\mathbf{y}(t)$, then $\mathbf{u}(t-\tau)$ results in output $\mathbf{y}(t-\tau)$.

We can show an example from circuit theory. Let $u(t)$ be a voltage applied to an RLC circuit resulting in a current $i(t)$. See Figure 1.1.

The basic relations governing voltage and current for resistors, inductors and capacitors are

$$v_L = L\frac{di}{dt},$$

$$C\frac{dv_C}{dt} = i(t),$$

$$v_R = i(t)R.$$

By substitution we get

$$v_L = LC\frac{d^2v_C}{dt^2},$$

$$v_R = RC\frac{dv_C}{dt}.$$

Kirchoff's law states that $u(t) = v_R + v_L + v_C$, and therefore

$$u(t) = LC\frac{d^2 v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C.$$

If

$$\mathbf{x}(t) = \begin{bmatrix} v_C \\ \dot{v}_C \end{bmatrix}$$

then

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ \frac{-1}{LC} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} v_C \\ \dot{v}_C \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u(t).$$

For background on matrix equations and application to optimal control see [1] and [2].

Chapter 2

Background

Mathematical control theory is the study of the design of controls, which can force a system to achieve a certain state, or at least be bounded near a certain state. Control theory as a formal subject is relatively new. Some of the pioneering work in the field was done by Maxwell, Lyapunov, and Kalman.

In 1868, J.C. Maxwell studied the stability of an engine. He analyzed the characteristic polynomial of the system and established the link between the roots of the polynomial and the behavior of the system. In particular, he established the criterion that the system is stable if and only if the real parts of all roots are strictly negative.

In 1892, A.M. Lyapunov studied the stability of nonlinear differential equations. His work is considered the foundation of nonlinear analysis and modern control theory.

Modern control theory became well established in 1960, when R. Kalman and associates published three papers, which built upon the work of Lyapunov and introduced the linear quadratic regulator in optimal control design, as well as applying optimal filtering, including the discrete Kalman filter.

In order to be able to analyze linear control systems there are some concepts which need to be introduced.

(2.0.1) Definition. For a linear system of the form $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$, the *state transition matrix* is the matrix P, such that $\mathbf{x}(t) = P(t, t_0)\mathbf{x}(t_0)$, under a zero input.

(2.0.2) Proposition. *If* $\mathbf{x}(t_0) = \mathbf{x}_0$*, then* $\mathbf{x}(t) = P(t, t_0)\mathbf{x}_0$*, where* $P(t, t_0) = e^{A(t-t_0)}$.

Proof: We have

$$\dot{\mathbf{x}}(t) = \frac{d}{dt} e^{A(t-t_0)}\mathbf{x}(t_0) = Ae^{A(t-t_0)}\mathbf{x}(t_0) = A\mathbf{x}(t)$$

$$\mathbf{x}(t_0) = P(t_0, t_0)\mathbf{x}_0 = e^0\mathbf{x}_0 = I\mathbf{x}_0 = \mathbf{x}_0.$$

Therefore, $P(t, t_0)\mathbf{x}_0$ is the state by uniqueness of the solution. $\square$

(2.0.3) Definition. A linear time invariant system is said to be *stable*, if each eigenvalue λ of the matrix A satisfies the condition $\mathrm{Re}(\lambda) < 0$. In other words, the eigenvalues lie in the left open half of the complex plane.

(2.0.4) Definition. A system is said to be in a *steady state*, when the state is constant.

(2.0.5) Theorem. *For a particular input* $\mathbf{u}(t)$ *and initial state* $\mathbf{x}_0$*, the* $\mathbf{x}(t)$*, such that*

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

with initial condition $\mathbf{x}(0) = \mathbf{x}_0$*, is given by*

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)\, d\tau.$$

Proof:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= Ae^{At}\mathbf{x}_0 + B\mathbf{u}(t) + A\int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)\,d\tau \\ &= A\mathbf{x}(t) + B\mathbf{u}(t)\end{aligned}$$

with initial condition

$$\mathbf{x}(0) = e^0\mathbf{x}_0 + \int_0^0 e^{A(0-\tau)}B\mathbf{u}(\tau)\,d\tau = \mathbf{x}_0.$$

□

(2.0.6) Definition. A system of the form (1.1) is *controllable* if for any $t_1 > t_0$ and for any $\mathbf{x}_0$ and $\mathbf{x}_1$ there exists an input $\mathbf{u}(t)$ such that the system is taken from initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ to final state $\mathbf{x}(t_1) = \mathbf{x}_1$.

There are two important matrices related to controllability.

(2.0.7) Definition. For $t > 0$ we define the *controllability grammian* $P(A,B)(t)$ by

$$P(A,B)(t) = \int_0^t e^{A\tau}BB^{\mathrm{T}}e^{A^{\mathrm{T}}\tau}\,d\tau.$$

(2.0.8) Definition. For a system of the form (1.1) we define the *controllability matrix*

$$C(A,B) = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}.$$

These matrices are related by the following Lemma.

(2.0.9) Lemma. *The* controllability matrix $C(A,B)$ *has rank* n *if and only if* $P(A,B)(t)$ *is positive definite, for all* $t > 0$.

Proof: We first note that the controllability grammian is at least positive semidefinite. Suppose that $C(A,B)$ has rank less than n. Then there exists a nonzero vector $\mathbf{x}_0$ such that

$\mathbf{x}_0^{\mathrm{T}} C(A,B) = 0$. This implies that $\mathbf{x}_0^{\mathrm{T}} A^k B = 0$ for $k \geq 0$. For $0 \leq k \leq n-1$, this is obvious. For $k \geq n$ it follows from the Cayley-Hamilton theorem that $A^k = p(A)$ where $p(A)$ is a degree $n-1$ polynomial in A. Thus

$$\mathbf{x}_0^{\mathrm{T}} A^k B = \mathbf{x}_0^{\mathrm{T}} p(A) B = 0.$$

This in turn means that $\mathbf{x}_0^{\mathrm{T}} e^{At} B = 0$ for $t \geq 0$. Therefore, it follows that

$$\int_0^t \mathbf{x}_0^{\mathrm{T}} e^{A\tau} B B^{\mathrm{T}} e^{A^{\mathrm{T}}\tau} \mathbf{x}_0 \, d\tau = 0$$

for $t > 0$. Thus $P(A,B)(t)$ is not positive definite.

Now suppose that there exists $T > 0$ so that $P(A,B)(T)$ is not positive definite. Then there exists $\mathbf{x}_0 \neq 0$ such that

$$\int_0^T \mathbf{x}_0^{\mathrm{T}} e^{A\tau} B B^{\mathrm{T}} e^{A^{\mathrm{T}}\tau} \mathbf{x}_0 \, d\tau = \int_0^T \| B^{\mathrm{T}} e^{A^{\mathrm{T}}\tau} \mathbf{x}_0 \|_2^2 \, d\tau = 0$$

so that $B^{\mathrm{T}} e^{A^{\mathrm{T}} t} \mathbf{x}_0 = 0$, for $t \geq 0$.

Transposing and differentiating this $n-1$ times with respect to t and evaluating at $t = 0$ gives

$$\mathbf{x}_0^{\mathrm{T}} B = \mathbf{x}_0^{\mathrm{T}} A B = \cdots = \mathbf{x}_0^{\mathrm{T}} A^{n-1} B = 0.$$

Thus the $n \times mn$ matrix $C(A,B)$ has rank less than n. We have used the invertibility of e^{At}. □

There are multiple tests for controllability. In particular

(2.0.10) Theorem. *A system is controllable if and only if the* controllability matrix *$C(A,B)$ has rank n (or equivalently the controllability grammian is positive definite).*

Proof: If $C(A,B)$ has rank n, then by Lemma (2.0.9) $P(A,B)(t)$ is positive definite and thus invertible. We define the mapping of an input $\mathbf{u}(t)$ to its corresponding resulting state $\mathbf{x}(t_1)$ at $t_1 > 0$ by

$$\mathbf{x}(t_1) = e^{A(t_1-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t_1} e^{A(t_1-\tau)}B\mathbf{u}(\tau)\,d\tau.$$

If we substitute a control input defined by

$$\mathbf{u}(t) = -B^{\mathrm{T}}e^{A^{\mathrm{T}}(t_1-t)}P(A,B)^{-1}(t_1-t_0)(e^{A(t_1-t_0)}\mathbf{x}_0 - \mathbf{x}_1)$$

into the mapping, for some initial state $\mathbf{x}_0$ and desired final state $\mathbf{x}_1$, then we obtain

$$\mathbf{x}(t_1) = e^{A(t_1-t_0)}\mathbf{x}(t_0) - \left(\int_{t_0}^{t_1} e^{A(t_1-\tau)}BB^{\mathrm{T}}e^{A^{\mathrm{T}}(t_1-\tau)}\,d\tau\right)P(A,B)^{-1}(t_1-t_0)(e^{A(t_1-t_0)}\mathbf{x}_0 - \mathbf{x}_1).$$

By the method of substitution within the integral, defining $v = t_1 - \tau$, the mapping becomes

$$\mathbf{x}(t_1) = e^{A(t_1-t_0)}\mathbf{x}(t_0) + \left(\int_{t_1-t_0}^{0} e^{Av}BB^{\mathrm{T}}e^{A^{\mathrm{T}}v}\,dv\right)P(A,B)^{-1}(t_1-t_0)(e^{A(t_1-t_0)}\mathbf{x}_0 - \mathbf{x}_1).$$

Then reversing the limits of integration gives

$$\mathbf{x}(t_1) = e^{A(t_1-t_0)}\mathbf{x}(t_0) - \left(\int_{0}^{t_1-t_0} e^{Av}BB^{\mathrm{T}}e^{A^{\mathrm{T}}v}\,dv\right)P(A,B)^{-1}(t_1-t_0)(e^{A(t_1-t_0)}\mathbf{x}_0 - \mathbf{x}_1).$$

The integral is simply the controllability grammian, so we now have

$$\begin{aligned}\mathbf{x}(t_1) &= e^{A(t_1-t_0)}\mathbf{x}(t_0) - P(A,B)(t_1-t_0)P(A,B)^{-1}(t_1-t_0)(e^{A(t_1-t_0)}\mathbf{x}_0 - \mathbf{x}_1)\\ &= e^{\mathbf{A}(t_1-t_0)}\mathbf{x}(t_0) - (e^{\mathbf{A}(t_1-t_0)}\mathbf{x}_0 - \mathbf{x}_1)\\ &= \mathbf{x}_1.\end{aligned}$$

thus the system is controllable.

Now suppose that $C(A, B)$ has rank less than n. Then the controllability grammian is positive semidefinite for some $T > 0$. There exists some $\mathbf{x}_1 \neq 0$ such that

$$\mathbf{x}_1^{\mathrm{T}} P(A, B)(T)\mathbf{x}_1 = 0.$$

Therefore

$$\int_0^T \mathbf{x}_1^{\mathrm{T}} e^{At} B B^{\mathrm{T}} e^{A^{\mathrm{T}} t} \mathbf{x}_1 \, dt = 0,$$

$$\int_0^T \left(B^{\mathrm{T}} e^{A^{\mathrm{T}} t} \mathbf{x}_1 \right)^{\mathrm{T}} \left(B^{\mathrm{T}} e^{A^{\mathrm{T}} t} \mathbf{x}_1 \right) \, dt = 0,$$

and

$$\int_0^T \| B^{\mathrm{T}} e^{A^{\mathrm{T}} t} \mathbf{x}_1 \|_2^2 \, dt = 0.$$

This implies $\mathbf{x}_1^{\mathrm{T}} e^{At} B = 0$, for all $t \epsilon \, [0, T]$.

Let the initial condition be $\mathbf{x}_0 = e^{-AT}\mathbf{x}_1$. Then

$$\begin{aligned} \mathbf{x}(T) &= e^{AT}\mathbf{x}_0 + \int_0^T e^{A(T-\tau)} B\mathbf{u}(\tau) \, d\tau \\ &= e^{AT} e^{-AT}\mathbf{x}_1 + \int_0^T e^{A(T-\tau)} B\mathbf{u}(\tau) \, d\tau \\ &= \mathbf{x}_1 + \int_0^T e^{A(T-\tau)} B\mathbf{u}(\tau) \, d\tau. \end{aligned}$$

Multiplying by $\mathbf{x}_1^{\mathrm{T}}$ gives

$$\mathbf{x}_1^{\mathrm{T}}\mathbf{x}(T) = \mathbf{x}_1^{\mathrm{T}}\mathbf{x}_1 + \int_0^T \mathbf{x}_1^{\mathrm{T}} e^{A(T-\tau)} B\mathbf{u}(\tau) \, d\tau.$$

If $\mathbf{x}(T) = 0$, then

$$0 = \| \mathbf{x}_1 \|^2 + \int_0^T 0\mathbf{u}(\tau) \, d\tau = \| \mathbf{x}_1 \|^2,$$

which contradicts $\mathbf{x}_1 \neq 0$. Thus there is no $\mathbf{u}(t)$ which brings $\mathbf{x}(T)$ to zero and the system is not controllable. □

(2.0.11) Definition. A system of the form (1.1) is *observable* if for any $t_1 > t_0$, the state $\mathbf{x}(t_1)$ can be known with only knowledge of the input $\mathbf{u}(t)$, the output $\mathbf{y}(t)$, and the initial state $\mathbf{x}(t_0)$.

(2.0.12) Definition. For a system of the form (1.1) we define the *observability matrix*

$$O(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

(2.0.13) Theorem. *A system is observable if and only if the* observability matrix $O(A, C)$ *has rank* n.

The proof is the dual of that for the theorem relating controllability to rank.

(2.0.14) Definition. A system is said to be *minimal*, if it is both controllable and observable.

Controllability and Observability can also be characterized in another form known as the Popov-Belevitch-Hautus test. This test states that a system is controllable if the matrix

$$\begin{bmatrix} A - \lambda I & B \end{bmatrix}$$

has rank n for all λ, eigenvalues of matrix A.

This test also states that a system is observable if the matrix

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$$

has rank n for all λ, eigenvalues of matrix A.

For background on control theory see [2].

Chapter 3

Solution

Our main concern is to have control over the system of the form (1.1). Control will allow the system to converge towards a desired state. Control is handled through a feedback input, which depends upon the state of the system. We wish to choose an input to drive the state of the system toward zero while also limiting the size of the input $\mathbf{u}(t)$. More precisely we wish to minimize

$$J = \int_{t_0}^{t_1} \left[\mathbf{x}^{\mathrm{T}}(t)Q\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)R\mathbf{u}(t)\right]\, dt + \mathbf{x}^{\mathrm{T}}(t_1)Z_1\mathbf{x}(t_1),$$

where Q is positive semi-definite and represents the cost penalty attributed to the state, R is positive definite and represents the cost penalty attributed to the input, and Z_1 is positive definite and represents the cost penalty attributed to the final state.

We can represent the input as a matrix product for some vector, $\mathbf{p}(t)$, as

$$\mathbf{u}(t) = -R^{-1}B^{\mathrm{T}}\mathbf{p}(t).$$

This is always possible if B has linearly independent columns.

Let $S = BR^{-1}B^{\mathrm{T}}$. Then the state equation becomes

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) - S\mathbf{p}(t)$$

and the cost function, J, becomes

$$J = \int_{t_0}^{t_1} \left[\mathbf{x}^{\mathrm{T}}(t)Q\mathbf{x}(t) + \mathbf{p}^{\mathrm{T}}(t)S\mathbf{p}(t)\right]\, dt + \mathbf{x}^{\mathrm{T}}(t_1)Z_1\mathbf{x}(t_1).$$

We wish to choose $\mathbf{p}(t)$ to minimize J. This can be accomplished by recognizing the fact that for some chosen input $\mathbf{p}(t)+\delta\hat{\mathbf{p}}(t)$, which is some deviation from the desired stabilizing input, the corresponding state is (by linearity) $\mathbf{x}(t)+\delta\hat{\mathbf{x}}(t)$. Here $\hat{\mathbf{x}}(t)$ is the state that results from applying input $\hat{\mathbf{p}}(t)$ to the state equation with initial condition $\hat{\mathbf{x}}(t_0) = 0$. The adjusted cost function now becomes

$$\begin{aligned}\hat{J} \;\; = \;\; & \int_{t_0}^{t_1} [(\mathbf{x}(t) + \delta\hat{\mathbf{x}}(t))^{\mathrm{T}}Q(\mathbf{x}(t) + \delta\hat{\mathbf{x}}(t)) + (\mathbf{p}(t) + \delta\hat{\mathbf{p}}(t))^{\mathrm{T}}S(\mathbf{p}(t) + \delta\hat{\mathbf{p}}(t))]dt \\ & +(\mathbf{x}(t_1) + \delta\hat{\mathbf{x}}(t_1))^{\mathrm{T}}Z_1(\mathbf{x}(t_1) + \delta\hat{\mathbf{x}}(t_1)).\end{aligned}$$

It can be shown that J is a minimum [3] when the derivative of $\hat{J}$ with respect to δ is zero. Taking the derivative with respect to δ gives

$$\left.\frac{d\hat{J}}{d\delta}\right|_{\delta=0} = 2\int_{t_0}^{t_1} \left[\mathbf{x}^{\mathrm{T}}(t)Q\hat{\mathbf{x}}(t) + \mathbf{p}^{\mathrm{T}}(t)S\hat{\mathbf{p}}(t)\right]\, dt + 2\mathbf{x}^{\mathrm{T}}(t_1)Z_1\hat{\mathbf{x}}(t_1).$$

Now using the modified state equation, we can substitute $S\hat{\mathbf{p}}(t) = A\hat{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t)$ and set the derivative equal to zero.

$$0 = \int_{t_0}^{t_1} \left[\mathbf{x}^{\mathrm{T}}(t)\, Q\hat{\mathbf{x}}(t) + \mathbf{p}^{\mathrm{T}}(t)A\hat{\mathbf{x}}(t) - \mathbf{p}^{\mathrm{T}}(t)\, \dot{\hat{\mathbf{x}}}(t)\right]\, dt + \mathbf{x}^{\mathrm{T}}(t_1)\, Z_1\hat{\mathbf{x}}(t_1).$$

Using integration by parts, we can substitute

$$\int_{t_0}^{t_1} \mathbf{p}^{\mathrm{T}}(t)\,\dot{\hat{\mathbf{x}}}(t)\,dt = \mathbf{p}^{\mathrm{T}}(t_1)\,\hat{\mathbf{x}}(t_1) - \int_{t_0}^{t_1} \dot{\mathbf{p}}^{\mathrm{T}}(t)\,\hat{\mathbf{x}}(t)dt.$$

Then using $\hat{\mathbf{x}}(t_0) = 0$, our equation becomes

$$0 = \int_{t_0}^{t_1} \left[\mathbf{x}^{\mathrm{T}}(t)\,Q + \mathbf{p}^{\mathrm{T}}(t)A + \dot{\mathbf{p}}^{\mathrm{T}}(t)\right]\,\hat{\mathbf{x}}(t)\,dt + \left[\mathbf{x}^{\mathrm{T}}(t_1)\,Z_1 - \mathbf{p}^{\mathrm{T}}(t_1)\right]\,\hat{\mathbf{x}}(t_1).$$

If the system is controllable, then a suitable choice of $\hat{\mathbf{p}}(t)$ gives any $\hat{\mathbf{x}}(t)$. This leads to the following requirements for our derivative to equal zero:

$$\mathbf{x}^{\mathrm{T}}(t)\,Q + \mathbf{p}^{\mathrm{T}}(t)\,A + \dot{\mathbf{p}}^{\mathrm{T}}(t) = 0$$

and

$$\mathbf{x}^{\mathrm{T}}(t_1)\,Z_1 = \mathbf{p}^{\mathrm{T}}(t_1).$$

Now the system becomes

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) - S\mathbf{p}(t)$$

with initial condition $\mathbf{x}(t_0) = x_0$

$$\dot{\mathbf{p}}(t) = -Q\mathbf{x}(t) - A^{\mathrm{T}}\mathbf{p}(t)$$

with final condition $\mathbf{p}(t_1) = Z_1\mathbf{x}(t_1)$.

This can be represented as a matrix differential system

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{p}(t) \end{bmatrix},$$

where

$$H = \begin{bmatrix} A & -S \\ -Q & -A^{\mathrm{T}} \end{bmatrix}$$

is a *Hamiltonian matrix* and initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ and final condition $\mathbf{p}(t_1) = Z_1\mathbf{x}(t_1)$.

By linearity there exists a state transition matrix

$$P(t, t_1) = \begin{bmatrix} P_{11}(t, t_1) & P_{12}(t, t_1) \\ P_{21}(t, t_1) & P_{22}(t, t_1) \end{bmatrix}$$

such that

$$\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{p}(t) \end{bmatrix} = \begin{bmatrix} P_{11}(t, t_1) & P_{12}(t, t_1) \\ P_{21}(t, t_1) & P_{22}(t, t_1) \end{bmatrix} \begin{bmatrix} \mathbf{x}(t_1) \\ \mathbf{p}(t_1) \end{bmatrix}.$$

However, taking derivatives of this matrix equation gives

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \dot{P}_{11}(t, t_1) & \dot{P}_{12}(t, t_1) \\ \dot{P}_{21}(t, t_1) & \dot{P}_{22}(t, t_1) \end{bmatrix} \begin{bmatrix} \mathbf{x}(t_1) \\ \mathbf{p}(t_1) \end{bmatrix}.$$

Then the matrix differential system becomes

$$\begin{bmatrix} \dot{P}_{11}(t, t_1) & \dot{P}_{12}(t, t_1) \\ \dot{P}_{21}(t, t_1) & \dot{P}_{22}(t, t_1) \end{bmatrix} \begin{bmatrix} \mathbf{x}(t_1) \\ \mathbf{p}(t_1) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} P_{11}(t, t_1) & P_{12}(t, t_1) \\ P_{21}(t, t_1) & P_{22}(t, t_1) \end{bmatrix} \begin{bmatrix} \mathbf{x}(t_1) \\ \mathbf{p}(t_1) \end{bmatrix}.$$

The latter equation is true for all $\mathbf{p}(t_1)$and $\mathbf{x}(t_1)$, and implies that

$$\begin{bmatrix} \dot{P}_{11}(t, t_1) & \dot{P}_{12}(t, t_1) \\ \dot{P}_{21}(t, t_1) & \dot{P}_{22}(t, t_1) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} P_{11}(t, t_1) & P_{12}(t, t_1) \\ P_{21}(t, t_1) & P_{22}(t, t_1) \end{bmatrix}$$

or simply $\dot{P}(t, t_1) = \mathrm{H}P(t, t_1)$.

Recall that our final condition $\mathbf{p}(t_1) = Z_1\mathbf{x}(t_1)$ can be used to show

$$\mathbf{p}(t) = P_{21}(t,t_1)\mathbf{x}(t_1) + P_{22}(t,t_1)\mathbf{p}(t_1) = P_{21}(t,t_1)\mathbf{x}(t_1) + P_{22}(t,t_1)Z_1\mathbf{x}(t_1)$$

so that

$$\begin{aligned}\mathbf{x}(t) &= P_{11}(t,t_1)\mathbf{x}(t_1) + P_{12}(t,t_1)\mathbf{p}(t_1) = P_{11}(t,t_1)\mathbf{x}(t_1) + P_{12}(t,t_1)Z_1\mathbf{x}(t_1) \\ &= \left[P_{11}(t,t_1) + P_{12}(t,t_1)Z_1\right]\mathbf{x}(t_1).\end{aligned}$$

Thus

$$\mathbf{x}(t_1) = \left[P_{11}(t,t_1) + P_{12}(t,t_1)Z_1\right]^{-1}\mathbf{x}(t).$$

For comments on the invertibility of $P_{11}(t,t_1) + P_{12}(t,t_1)Z_1$ see Appendix 2.

Substituting this back into the equation for $\mathbf{p}(t)$, we obtain

$$\mathbf{p}(t) = \left[P_{21}(t,t_1) + P_{22}(t,t_1)Z_1\right]\left[P_{11}(t,t_1) + P_{12}(t,t_1)Z_1\right]^{-1}\mathbf{x}(t)$$

$$= \left[N(t)\right]\left[M(t)\right]^{-1}\mathbf{x}(t).$$

We can define

$$P(t) = \left[N(t)\right]\left[M(t)\right]^{-1},$$

which gives $\mathbf{p}(t) = P(t)\mathbf{x}(t)$.

By substituting, we obtain the following equation

$$\begin{bmatrix} \dot{M}(t) \\ \dot{N}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} M(t) \\ N(t) \end{bmatrix}.$$

Now our problem can be stated as finding the matrix $P(t)$, which through the previous processes will determine our solution to the optimization problem of our feedback for stability. It turns out that the solution $P(t)$ is also the solution to a *continuous time differential Riccati equation*

$$-\frac{dP(t)}{dt} = A^T P(t) + P(t)A - P(t)SP(t) + Q.$$

We can show this by taking the derivative of

$$P(t) = [N(t)]\,[M(t)]^{-1}$$

to get

$$\frac{dP(t)}{dt} = \frac{d\,[N(t)]}{dt}\,[M(t)]^{-1} + [N(t)]\,\frac{d\,[M(t)]^{-1}}{dt}.$$

Using the fact that $[M(t)]\,[M(t)]^{-1} = I$,we get

$$\frac{dI}{dt} = 0 = [M(t)]\,\frac{d\,[M(t)]^{-1}}{dt} + \frac{d\,[M(t)]}{dt}\,[M(t)]^{-1}\,.$$

We then obtain

$$\frac{d\,[M(t)]^{-1}}{dt} = -\,[M(t)]^{-1}\,\frac{d\,[M(t)]}{dt}\,[M(t)]^{-1}\,.$$

By substituting the above derivative into our solution, we get

$$\begin{aligned}\frac{dP(t)}{dt} &= \left(-QM(t) - A^{\mathrm{T}}N(t)\right)M^{-1}(t) - N(t)M^{-1}(t)\left(AM(t) - SN(t)\right)M^{-1}(t)\\ &= -Q - A^{\mathrm{T}}P(t) - P(t)A + P(t)SP(t)\end{aligned}$$

which is the Differential Riccati equation for our solution $P(t)$.

We desire a steady state stabilizing solution. We can do this by first finding an infinite horizon steady state solution P_∞ where $\dot{P}(t) = 0$. This can be shown to be optimal as $t_1 \to \infty$. See [1].

If $\dot{P}(t) = 0$ then the differential Riccati equation becomes

$$0 = A^T P_\infty + P_\infty A - P_\infty S P_\infty + Q.$$

It can be shown that $\hat{A} = A - SP_\infty$ is stable. This is called the *algebraic Riccati equation*.

The optimal choice of input that minimizes the infinite horizon cost function

$$J = \int_{t_0}^{\infty} \mathbf{x}^{\mathrm{T}}(t) Q \mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t) R \mathbf{u}(t)\, dt$$

is

$$\mathbf{u}(t) = -R^{-1} B^{\mathrm{T}} P_\infty \mathbf{x}(t).$$

Chapter 4

Solution of The Algebraic Riccati Equation

The Algebraic Riccati Equation has the form

$$0 = A^{\mathrm{T}}P + PA - PSP + Q,$$

where the matrices A,S, and Q are $n \times n$ matrices defined in terms of the system. It will be shown that such an equation has a solution P in terms of these system matrices.

We define H, a $2n \times 2n$ Hamiltonian matrix for our system as

$$H = \begin{bmatrix} A & -S \\ -Q & -A^{\mathrm{T}} \end{bmatrix}.$$

Suppose we have an invariant subspace $\mathcal{M}$ of H, where $H\mathbf{x}\epsilon\mathcal{M}$, if $\mathbf{x}\epsilon\mathcal{M}$.

Suppose the columns of $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ form a basis for $\mathcal{M}$, the invariant subspace of H, where P_1 is $n \times n$. Then

$$H \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} B$$

for some $n \times n$ matrix B.

Assume P_1 is invertible. Then

$$\begin{bmatrix} A & -S \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} P_1^{-1} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} P_1^{-1} P_1 B P_1^{-1} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} P_1^{-1} \hat{B},$$

so that

$$\begin{bmatrix} A & -S \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I \\ P_2 P_1^{-1} \end{bmatrix} = \begin{bmatrix} I \\ P_2 P_1^{-1} \end{bmatrix} \hat{B} = \begin{bmatrix} \hat{B} \\ P_2 P_1^{-1} \hat{B} \end{bmatrix}.$$

By multiplying the matrices, we get two equations

$$AI - S P_2 P_1^{-1} = \hat{B}$$

$$-Q - A^{\mathrm{T}} P_2 P_1^{-1} = P_2 P_1^{-1} \hat{B}.$$

Substituting from the first into the second gives

$$-Q - A^{\mathrm{T}} P_2 P_1^{-1} = P_2 P_1^{-1} (A - S P_2 P_1^{-1})$$

$$A^{\mathrm{T}} P_2 P_1^{-1} + P_2 P_1^{-1} A - P_2 P_1^{-1} S P_2 P_1^{-1} + Q = 0.$$

Therefore, $P = P_2 P_1^{-1}$ is a solution to the algebraic Riccati equation.

We have shown that a solution arises when we have an invariant subspace, but now we need to show how to obtain an invariant subspace. To do so, perform the Schur factorization

$$H = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} Q_1^{\mathrm{T}} \\ Q_2^{\mathrm{T}} \end{bmatrix} = QTQ^{\mathrm{T}}.$$

The following theorem summarizes the invariant subspace approach to computing algebraic Riccati equation solutions.

(4.0.15) Theorem. *Suppose the pair (A, B) is controllable and the pair (Q, A) is observable. We assume that Q is positive semidefinite and $S = BR^{-1}B^{\mathrm{T}}$, where R is positive definite.*

1. *Then the $2n \times 2n$ Hamiltonian matrix*

$$H = \begin{bmatrix} A & -S \\ -Q & -A^{\mathrm{T}} \end{bmatrix}$$

   has no pure imaginary eigenvalues. If λ is an eigenvalue of H, then $-\lambda$ is also an eigenvalue of H. Thus H has n eigenvalues in the open left half plane and n in the open right half plane.

2. *If the $2n \times n$ matrix*

$$\begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix}$$

   has columns that comprise a basis for the invariant subspace of H associated with the n eigenvalues of H in the left half plane (the stable invariant subspace), then P_{11} is invertible and $P = P_{21}P_{11}^{-1}$ is a solution to the algebraic Riccati equation

$$A^{\mathrm{T}}P + PA - PSP + Q = 0.$$

Further, P is symmetric and positive definite. The input

$$\mathbf{u}(t) = -R^{-1}BP\mathbf{x}(t)$$

minimizes

$$J = \int_{t_0}^{\infty} \mathbf{x}^{\mathrm{T}}(t)Q\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)R\mathbf{u}(t)\,dt$$

subject to the constraint that $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$. The value of J achieved in this minimization is given by $J = \mathbf{x}^{\mathrm{T}}(t_0)P\mathbf{x}(t_0)$.

Proof: We have already shown how an invariant subspace (if P_{11} is invertible) leads to a solution to the algebraic Riccati equation. The easiest way to prove the rest of part 2 is to construct P as a limiting case of a solution of the Riccati differential equation and then show, after the fact, that it can also be obtained from the stable invariant subspace of the Hamiltonian matrix, [1]. A more algebraic proof is possible, [3], but it is significantly more work. Either way, the full proof is beyond the scope of this thesis.

For part 1 we let

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

It can be easily verified that $HJ = (HJ)^{\mathrm{T}}$. So

$$H\mathbf{x} = \lambda\mathbf{x}$$

implies

$$J^{\mathrm{T}}HJJ^{\mathrm{T}}\mathbf{x} = \lambda J^{\mathrm{T}}\mathbf{x}$$

or

$$H^{\mathrm{T}}(J^{\mathrm{T}}\mathbf{x}) = -\lambda(J^{\mathrm{T}}\mathbf{x}).$$

Thus $-\lambda$ is an eigenvalue of H^{T} and must be an eigenvalue of H.

To show that H has no imaginary eigenvalues suppose

$$H \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \tag{4.1}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are not both zero and $\lambda + \overline{\lambda} = 0$. Note that in general the eigenvector will be complex. We seek a contradiction. Equation 4.1 gives

$$A\mathbf{x}_1 - S\mathbf{x}_2 = \lambda\mathbf{x}_1$$

so that

$$\mathbf{x}_2^* S\mathbf{x}_2 = \mathbf{x}_2^* A\mathbf{x}_1 - \lambda\mathbf{x}_2^*\mathbf{x}_1.$$

Equation 4.1 also gives

$$-Q\mathbf{x}_1 - A^{\mathrm{T}}\mathbf{x}_2 = \lambda\mathbf{x}_2$$

from which we conclude $\mathbf{x}_2^* A = -(\mathbf{x}_1^* Q - \overline{\lambda}\mathbf{x}_2^*)$. Thus

$$\mathbf{x}_2^* S\mathbf{x}_2 = -(\mathbf{x}_1^* Q + \overline{\lambda}\mathbf{x}_2^*)\mathbf{x}_1 - \lambda\mathbf{x}_2^*\mathbf{x}_1 = -\mathbf{x}_1^* Q\mathbf{x}_1.$$

Since Q and S are positive semidefinite, this implies $S\mathbf{x}_2 = Q\mathbf{x}_1 = 0$. It follows that $A\mathbf{x}_1 = \lambda\mathbf{x}_1$. Thus

$$\begin{bmatrix} A - \lambda I \\ Q \end{bmatrix} \mathbf{x}_1 = 0.$$

If $\mathbf{x}_1 \neq 0$, then this contradicts observability of (Q, A) by the Popov-Belevitch-Hautus test. Similarly, if $\mathbf{x}_2 \neq 0$, then we can show that

$$\mathbf{x}_2^* \begin{bmatrix} S & A + \overline{\lambda} I \end{bmatrix} = 0$$

which contradicts the controllability of (A, S). □

For further background on Riccati equations see [3].

Chapter 5

Numerical Tests

Here we show two simple examples of systems and the resulting solutions. Consider a system of the form

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

where the matrix coefficients are

$$A = \begin{bmatrix} -0.995591312646866 & -1.249081404879689 \\ 0.320053945411928 & 1.163391509093344 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.216261377558112 \\ 2.120734989643097 \end{bmatrix}.$$

The cost penalty matrices are chosen to be

$$Q = \begin{bmatrix} 2.60795573245784 & 1.26610295835102 \\ 1.26610295835102 & 2.95448749241003 \end{bmatrix}$$

$$R = \begin{bmatrix} 0.00137941893917394 \end{bmatrix}.$$

The Schur factorization T of the Hamiltonian matrix H is

$$T = \begin{bmatrix} -94.232548478034 & -2813.831700634160 & 1705.595743596213 & 70.245210254770 \\ 0 & 94.232548478035 & -57.644133237705 & -5.220714752416 \\ 0 & 0 & -1.103439836974 & -1.629924392466 \\ 0 & 0 & 0 & 1.103439836975 \end{bmatrix}.$$

Picking the invariant subspace associated with eigenvalues -94.232548478034 and 94.232548478035 gives a solution to the Riccati equation

$$P_\infty = \begin{bmatrix} 0.968927950931493 & 0.098990583917009 \\ 2.737032485924418 & 0.279178419647246 \end{bmatrix}.$$

This matrix is not symmetric positive definite. This is as expected, since the eigenvalues of H were not chosen in the left half plane.

In order to optimize our cost function J, it was stated that the solution must be symmetric positive definite with eigenvalues in the left half complex plane. In order to achieve this, we can revise the computations above with a simple reordering of the diagonal matrix T, so that the negative real part eigenvalues are in the beginning columns and the positive real part eigenvalues are in the ending columns. Then we will obtain revised results

$$T = \begin{bmatrix} -94.232548478034 & -3.62405968693 & 57.759814319867 & -3290.636828732672 \\ 0 & -1.103439836974 & 1.851376139053 & -57.759814319869 \\ 0 & 0 & 1.103439836975 & -3.62405968692 \\ 0 & 0 & 0 & 94.232548478035 \end{bmatrix}.$$

This will result in a solution to the Riccati equation

$$P_\infty = \begin{bmatrix} 1.328499383109633 & 0.131678278128935 \\ 0.131678278128935 & 0.042332513557622 \end{bmatrix}.$$

This solution is symmetric positive definite. The eigenvalues of P turn out to be $\lambda_1 = 1.341842237428075$ and $\lambda_2 = 0.028989659239181$. This is the optimal solution for this problem and gives an optimal control of

$$\mathbf{u}(t) = -R^{-1}B^{\mathrm{T}}P_\infty\mathbf{x}(t).$$

Chapter 6

Conclusions

This thesis has developed an optimal solution for the continuous linear time-invariant system by finding the solution that minimizes the cost function J. This was done by solving a related algebraic Riccati equation. The procedure for solving the Riccati equation was shown. We have seen that the selected Hamiltonian eigenvalues must lie in the left open half plane in order to ensure the solution will optimally control the system. A numerical example was given, showing how to compute and verify the solution for a simple 2×2 system.

Bibliography

[1] Anderson, B.D.O., Moore, J.B., *Optimal Control: Linear Quadratic Methods*, Prentice-Hall, 1989.

[2] F. Fairman, *Linear Control Theory: The State Space Approach*, John Wiley and Sons, 1998.

[3] P. Lancaster and L. Rodman, *Algebraic Riccati Equations*, Oxford Science Publications, 1995.

APPENDIX 1

The following is the coding used to do the numerical tests.

```
n=2; m=1;
t1=1; t0=.5;
A=[-0.995591312646866,-1.249081404879689;
   0.320053945411928,1.163391509093344];
B=[-0.216261377558112;
   2.120734989643097];
Q=[2.60795573245784,1.26610295835102;
   1.26610295835102,2.95448749241003];
R= 0.00137941893917394;
%A=randn(n,n);
%B=randn(n,m);
%Q=randn(n,n); Q=Q'*Q;
%R=randn(m,m); R=R'*R;
S=B*(R\B');
H=[A, -S;-Q,-A'];
% Compute the real Schur form of the Hamiltonian
[U,T]=schur(H);
% Make it a complex Schur form for easy eigenvalue swapping.
for j=1:2*n-1
  if (abs(T(j+1,j))>1e-16)
    [V,D]=eig(T(j:j+1,j:j+1));
    [G,r]=qr(V);
    T(j:j+1,:)=G'*T(j:j+1,:);
    T(:,j:j+1)=T(:,j:j+1)*G;
```

```
    U(:,j:j+1)=U(:,j:j+1)*G;
  end
end
% Swap eigenvalues so that unstable eigenvalues are first
% To get a negative definite solution to the CARE.
% Comment out to get no sorting of eigenvalues.
% Currently this is set to give a positive definite solution
% to the Riccati equation.
for j=1:2*n
  for k=1:2*n-1
    if (real(T(k,k)) > real(T(k+1,k+1)))
      G=givens(T(k,k+1),T(k+1,k+1)-T(k,k));
      T(k:k+1,:)=G*T(k:k+1,:);
      T(:,k:k+1)=T(:,k:k+1)*G';
      U(:,k:k+1)=U(:,k:k+1)*G';
      T(k+1,k)=0;
    end
  end
end
% Form the maximal CARE solution.
G=U(1:2*n,1:n)/U(1:n,1:n);
P=G(n+1:2*n,:);
J=eye(2*n); J=J([[n+1:2*n],[1:n]],:);
J(1:n,n+1:2*n)=-J(1:n,n+1:2*n);
```

APPENDIX 2

We assume invertibility of matrices at various key points. The assumptions can be justified using existence of a solution $P(t)$ to the differential Riccati equation

$$-\dot{P}(t) = A^{\mathrm{T}}P + PA - PSP + Q, \qquad P(t_1) = Z_1.$$

Existence of a solution can be proven [1] and we take it for granted.

Consider the Hamiltonian differential equation

$$\begin{bmatrix} \dot{M}(t) \\ \dot{N}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} M(t) \\ N(t) \end{bmatrix} = H \begin{bmatrix} M(t) \\ N(t) \end{bmatrix}, \qquad \begin{bmatrix} M(t_1) \\ N(t_1) \end{bmatrix} = \begin{bmatrix} I \\ Z_1 \end{bmatrix}.$$

This has a unique solution

$$\begin{bmatrix} M(t) \\ N(t) \end{bmatrix} = e^{H(t-t_1)} \begin{bmatrix} I \\ Z_1 \end{bmatrix}$$

which, since matrix exponentials are invertible, clearly has rank n.

If $P(t)$ is a solution to the Riccati equation, then it is easy to verify that $M(t)$ satisfying the differential equation

$$\dot{M}(t) = (A - SP(t))M(t), \qquad M(t_1) = I$$

and $N(t)$ given by

$$N(t) = P(t)M(t)$$

satisfy the Hamiltonian equation. Thus the solution to the Hamiltonian equation has the form

$$\begin{bmatrix} M(t) \\ N(t) \end{bmatrix} = \begin{bmatrix} I \\ P(t) \end{bmatrix} M(t) = \begin{bmatrix} P_{11}(t,t_1) & P_{12}(t,t_1) \\ P_{21}(t,t_1) & P_{22}(t,t_1) \end{bmatrix} \begin{bmatrix} I \\ Z_1 \end{bmatrix} = \begin{bmatrix} P_{11}(t,t_1) + P_{12}(t,t_1)Z_1 \\ P_{21}(t,t_1) + P_{22}(t,t_1)Z_1 \end{bmatrix}.$$

This has rank n only if $M(t) = P_{11}(t,t_1) + P_{12}(t,t_1)Z_1$ is invertible.