INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

PROBLEM SET- 3 **AUTUMN 2016**

MATHEMATICS-I(MA10001) **2nd August 2016**

1. Determine the following limits if they exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$ (b) $\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$ (c) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^2+y^2}$

(d) $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2}$ (e) $\lim_{(x,y)\to(0,0)} \frac{e^{\frac{-1}{x^2+y^2}}}{x^4+y^4}$

(f) $\lim_{(x,y)\to(0,0)} (1+x^2y^2)^{\frac{-1}{x^2+y^2}}$ (Hints. $e = \lim_{t\to 0}(1+t)^{(1/t)}$)

(g) $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$ (Hints. put $y = mx$.) (h) $\lim_{(x,y)\to(0,0)} \frac{2x^2+y^2}{x^2+y^2}$ (Hints. put $y = mx$.)

(i) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$ (j) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.

(Hints. for (a), (b), (c), (e), (i) ,(j), put $x = rcos\theta, y = rsin\theta$.)

2. Using $\epsilon -\ \delta$ approach, show that the following:

(a) $\lim_{(x,y)\to(0,0)} (|x|+|y|) = 0$ (b) $\lim_{(x,y)\to(2,1)} (x^2+2x-y^2) = 7$ (c) $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = 0$

(d) $\lim_{(x,y)\to(1,1)} (x^2+2y) = 3$ (e) $\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin(\frac{1}{xy}) = 0$

(f) $\lim_{(x,y)\to(0,0)} \frac{(x+y)}{x^2+y^2+1} = 0$ (g) $\lim_{(x,y)\to(1,1)} (y^2+3x) = 4$ (h) $\lim_{(x,y)\to(1,2)} (2x-3y) = -4$

(i) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$ (j) $\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2} = \frac{5}{2}$.

3. Discuss the continuity of the following functions at the point (0,0).

(i) $f(x,y) = \frac{x^2y^2}{x^2+y^2};\ f(0,0) = 0$ (ii) $f(x,y) = \frac{xy}{x^2+y^2};\ f(0,0) = 0$ (Hint. put $y = mx$.)

(iii) $f(x,y) = \frac{x^3y^3}{x^2+y^2};\ f(0,0) = 0$ (iv) $f(x,y) = \frac{1}{x^2+y^2};\ f(0,0) = 0$

(v) $f(x,y) = \frac{x^4-y^4}{x^4+y^4};\ f(0,0) = 0$ (Hint. put $y = mx$.)

(vi) $f(x,y) = \frac{x^2y^2}{x^4+y^4};\ f(0,0) = 0$ (Hint. put $y = mx$.)

(vii) $f(x,y) = \frac{e^{xy}}{x^2+1};\ f(0,0) = 1$ (viii) $f(x,y) = \frac{x^3+y^3}{x^2+y^2};\ f(0,0) = 0$

(ix) $f(x,y) = \frac{x^2y^2}{x^3+y^3};\ f(0,0) = 0$ (Hint. put $y = mx$, and $y = -xe^x$.)

(x) $f(x,y) = \frac{2xy^2}{x^2+y^4};\ f(0,0) = 0$. (Hint. put $x = my^2$.)

(Hints. for (i), (iii), put $x = rcos\theta, y = rsin\theta$.)

4. Find the points of discontinuities of the function $f(x,y) = \dfrac{1}{sin^2\pi x + sin^2\pi y}$.

5. Where the function $f(x,y) = \dfrac{1}{sin\ \pi x} + \dfrac{1}{sin\ \pi y}$ is discontinuous ?

6. Is it possible to define the function $f(x,y) = \dfrac{2xy}{x^2+y^2}$ at $(0,0)$ such that the function is continuous ? (Hints. put $y = mx$.)

7. Let $f(x,y) = \left\{ \begin{array}{ll} 0, & \text{for } xy \neq 0 \\ 1, & \text{for } xy = 0 \end{array} \right\}$ (i) Find the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along the line $y = x$. (ii) Is $f(x,y)$ continuous at origin ?

8. Show that the function $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{|x|+|y|} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is continuous at $(0,0)$ but $f_x(0,0)$ and $f_y(0,0)$ does not exist.

9. Show that the function $f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is not continuous at $(0,0)$ but $f_x(0,0)$ and $f_y(0,0)$ exist. (Hints. put $y = mx$.)

10. Show that the function $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ has partial derivative at $(0,0)$ but partial derivatives are not continuous at $(0,0)$. (Hints. put $y = mx$.)

11. For the function $f(x,y) = \begin{cases} \dfrac{x^2y(x-y)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ show that $f_{xy} \neq f_{yx}$ at $(0,0)$.

Answers

Q. 1.
(a) Limit exists and value is 2. (b) Limit exists and value is 0. (c) Limit exists and value is 0.
(d) Limit exists and and value is infinity. (e) Limit exists and value is 0. (f) Limit exists and value is 1.
(g) Limit does not exist. (h) Limit does not exist. (i) Limit exists and value is 0. (j) Limit exists and value is 0.
Q. 3.
(i) Continuous. (ii) Limit does not exist. (iii) Continuous.
(iv) Discontinuous. (v) Limit does not exist. (vi) Limit does not exist.
(vii) Continuous. (viii) Continuous. (ix) Limit does not exist. (x) Limit does not exist.
Q. 4. $x = m, y = n : m, n \in \mathbb{Z}$.
Q. 5. $x = m, y = n; m, n \in \mathbb{Z}$.
Q. 6. No
Q. 7. (i) limit does not exist along the line $y = x$. (ii) not continuous at origin.

End