Problem Set - 1

AUTUMN 2016

MATHEMATICS-I (MA10001) *July 18, 2016*

1. If $f(x) = \tan x$, then $f(x)$ vanishes for $x = 0$ and $x = \pi$. Is Rolle's theorem is applicable to the function in $[0, \pi]$? Give justification. Also give an example to show that the conditions of Rolle's theorem are sufficient but not necessary.

2. Given the function $f(x) = 1 + x^m(x-1)^n$, where m, n are positive integers. Without computing the derivative, show that $f'(x) = 0$ has atleast one root in $(0, 1)$.

3. A function f is thrice differentiable on $[a, b]$ and $f(a) = 0 = f(b)$ and $f'(a) = 0 = f'(b)$. prove that $f'''(c) = 0$ for some $c \in (a, b)$.

4. If $P(x)$ is a polynomial and $k \in \mathbb{R}$, prove that between two real roots of $P(x) = 0$ there is a root of $P'(x) + kP(x) = 0$.

5. If $f(x)$ and $g(x)$ are continuous functions on $[a, b]$ and differentiable on (a, b), then show that

$$f(a)g(b) - g(a)f(b) = (b-a)\{f(a)g'(c) - g(a)f'(c)\}$$

where $a < c < b$.

6. Prove that the function $f(x) = x^n + px + q$ cannot have more than two real roots if n is even and more than three if n is odd.

7. Prove that if the equation $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x = 0$ has a positive root x_0, then the equation $na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots + a_{n-1} = 0$ has a positive root less than x_0.

8. Suppose the functions $f(x), g(x)$ and their derivatives $f'(x), g'(x)$ are continuous through out a certain interval and $f(x)g'(x) - g(x)f'(x)$ never vanishes at any point of this interval. Show that between any two real roots of $f(x) = 0$, there lies one of the root of $g(x) = 0$ and conversely.

9. Prove that the equation $x^4 - 4x - 1 = 0$ has two different real roots.

10. Show that the equation $x^3 - 3x + k = 0\,(k \in \mathbb{R})$ can't have two distinct roots in the interval $(0, 1)$.

11. Without finding the derivative prove that all roots of the derivative of the given function $f(x) = (x+1)(x-1)(x-2)(x-3)$ are real.

12. If $c_0 + \dfrac{c_1}{2} + \ldots + \dfrac{c_{n-1}}{n} + \dfrac{c_n}{n+1} = 0$, where $c_0, c_1, \ldots, c_n$ are real constant, show that the equation $c_0 + c_1x + \ldots + c_nx^{n-1} + c_nx^n = 0$ has at leat one real root between 0 and 1.

13. Use mean value theorem to prove

    (a) $0 < \frac{1}{x}\ln\frac{e^x-1}{x} < 1$, for $x > 0$

    (b) $\frac{a-b}{\cos^2 b} \leq \tan a - \tan b \leq \frac{a-b}{\cos^2 a}$, for $0 < b \leq a < \frac{\pi}{2}$

(c) $\frac{x}{1+x^2} < \arctan x < x$ for $x > 0$ (d) $nb^{n-1}(a-b) < a^n - b^n < na^{n-1}(a-b)$, $a > b$

(e) $\frac{a-b}{a} \leq \ln\frac{a}{b} \leq \frac{a-b}{b}$ for $0 < b \leq a$. (f) $\frac{a-b}{1+a^2} < \arctan(a) - \arctan(b) < \frac{a-b}{1+b^2}$, $0 < b < a$

14. If $f'(x), g'(x)$ are continuous in $[a, b]$ and differentiable in (a, b) then show that there exists $c, a < c < b$ s.t.

$$\frac{f(b) - f(a) - (b-a)f'(a)}{g(b) - g(a) - (b-a)g'(a)} = \frac{f''(c)}{g''(c)}.$$

15. Prove the following results:

    (a) If $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = \sqrt{x}$, then c is the geometric mean between a and b; $a < c < b$.

    (b) If $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$, then c is the harmonic mean between a and b; $a < c < b$.

16. If $\phi(x)$ and $\psi(x)$ are continuous for $a \leq x \leq b$ differentiable for $a < x < b$, and $\psi'(x)$ never vanishes, there for some ξ in (a, b)

$$\frac{\phi(\xi) - \phi(a)}{\psi(b) - \psi(\xi)} = \frac{\phi'(\xi)}{\psi'(\xi)}.$$

17. Let $g(x) = 2f(\frac{x}{2}) + f(2-x)$ and $f''(x) < 0, \forall x \in (0, 2)$. Find the intervals of increase and decrease of $g(x)$.

18. Determine the intervals in which the function $f(x) = (x^4 + 6x^3 + 17x^2 + 32x + 32)e^{-x}$ is increasing or decreasing.

19. Show that

    (a) $x/\sin x$ increases in the interval $]0, \pi/2[$.

    (b) $x/\tan x$ decreases in the interval $]0, \pi/2[$.

20. If f is differentiable on $[0, 1]$, show by Cauchy mean value theorem that $f(1) - f(0) = \frac{f'(x)}{2x}$ has at least one solution in $(0, 1)$.

21. Write the Cauchy mean value theorem formula for the function $f(x) = x^2, \phi(x) = x^3$ on the interval $[1, 2]$ and find c.