Lecture-9

Probability & Statistics.

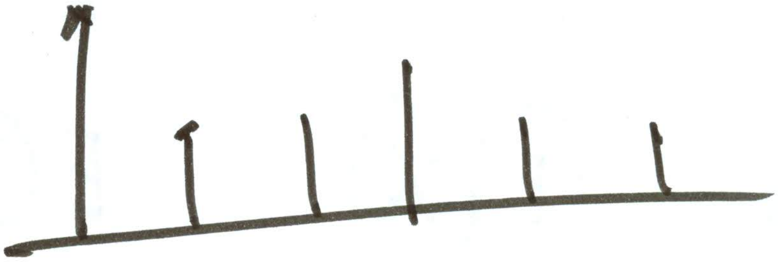
(mean)
Expectation, variance, SD,
n-th moment.

Given a discrete random variable

$E[X]$, $var(X)$, $\sqrt{var(X)}$,

$E[X^n]$.

$X(w) \in \mathbb{R}$



$f(x_i) \neq 0$, $i = 1, 2, \dots$

$$\sum_i x_i f(x_i) = \sum_i x_i P(X = x_i)$$

$$x_1, x_2, \dots, x_n \longrightarrow \frac{x_1 + x_2 + \dots + x_n}{n}$$

$$= \frac{1}{n} x_1 + \frac{1}{n} x_2 + \dots + \frac{1}{n} x_n$$

weighted avg.

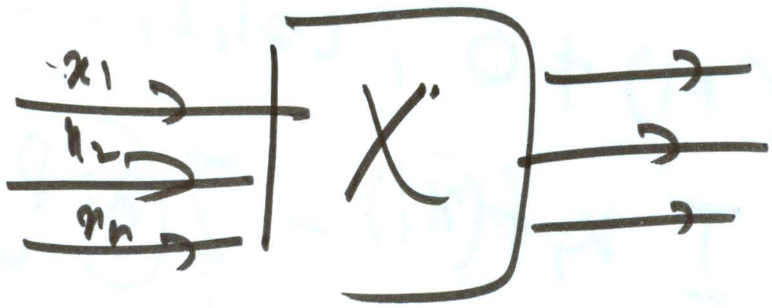
$$\hookrightarrow = \sum_i x_i p_i \,, \quad p_i \geq 0$$

$$\sum_i p_i = 1 .$$

$$x_1, x_2, \ldots, x_n, \ldots$$

Expectation = weighted avg. of x_is and the weights are probabilities.

$E(X)$ $\quad E(Y)$ $\quad + : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$



$$x = \sum \alpha_i x_i$$

$$X(x) = \, ? \, \sum \alpha_i X(x_1)$$

Prob 1.

$$E[\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n] = \alpha_1 E[X_1] + \alpha_2 E[X_2] + \cdots + \alpha_n E[X_n]$$

It is sufficient to show

$$\begin{cases} E[X+Y] = \underline{E[X]} + E[Y] \\ E[cX] = c\,E[X] \end{cases}$$

$$E[X] + E[Y] = \sum_i x_i P(X = x_i) + \sum_j y_j \underline{P(Y = y_j)}$$

$$= \sum_i x_i \left[\sum_j P(\{X = x_i\} \cap \{Y = y_j\}) \right] + \sum_j y_j \left[\sum_i P(\{Y = y_j\} \cap \{X = x_i\}) \right]$$

$$= \sum_i \sum_j x_i \, P(-) + \sum_j \sum_i y_j \, P(-)$$

$$= \sum_i \sum_j (x_i + y_j) \, P(-)$$

$$= E[X+Y].$$

$$E[cX] = \sum_i x_i \, P(cX = x_i)$$

$$= c \sum (x_i/c) \, P(X = x_i/c)$$

wt $c \neq 0$

$$= c \sum k \, P(X = k)$$

$$= c \, E[X].$$

Geometric Distribution.

X = # of trials untill the first success occurs (including the success)

$$P(X=k) = (1-p)^{k-1}p.$$

$k = 1, 2, \ldots$

Problem. Let $k > i$

$$P(X=k \mid X>i)$$

$$= \frac{P(\{X=k\} \cap \{X>i\})}{P(X>i)}$$

$$= \frac{P(X=k)}{P(X>i)} = \frac{(1-p)^{k-1}p}{(1-p)^i}$$

$$\boxed{P(X>i) = (1-p)^i} = (1-p)^{k-1-i}p$$

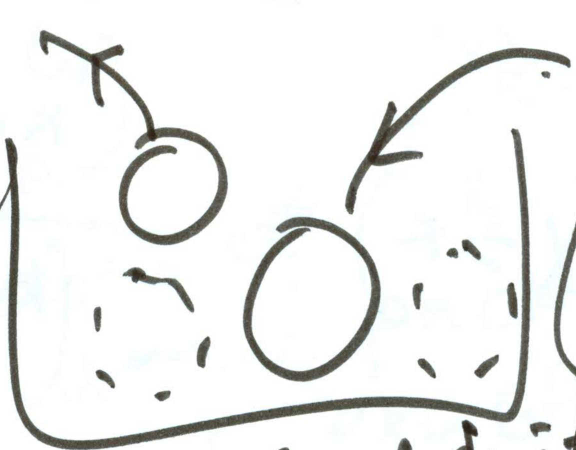
$$= (1-p)^{(k-i)-1}p.$$

Memoryless property of a random variable.

Def. Suppose $c > 0$

and $P(X = k + c \mid X > k) = P(X = c)$

then the r.v. X has the memoryless property.



$\text{Bin}\left(n, \frac{w}{w+b}\right)$

X = the # of white balls obtained in n trials.

Story.

Consider an urn with w white balls and b black balls. We draw n balls out of the urn at random without replacement, such that $\binom{w+b}{n}$ samples are equally likely.

Let X = the # of white balls in the sample.

Then $$P(X=k) = \frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$$

To get $P(X=k)$, first count the # of possible ways to draw exactly k white balls and $n-k$ black balls from the urn (without distinguishing between different orderings for getting the same set of balls.

'hyper' – more than.

Q. Calculate the prob. that a hand of 13 cards from a standard pack of 52 cards contains

1. exactly 5 clubs
2. 5 clubs or 5 spades or both

Ans. 1.

$$\frac{\binom{13}{5}\binom{39}{8}}{\binom{52}{13}}$$

2.

$$\frac{2\binom{13}{5}\binom{39}{8} - \binom{13}{5}\binom{13}{5}\binom{26}{3}}{\binom{52}{13}}$$

— 5 clubs (event A)

— 5 spads (event B)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

?