Lecture-8

Probability & Statistics.

Discrete random variable (X)

- → PMF / prob. density function
- → PDF (probability distribution function)

$X : \Omega \to \mathbb{R}$

$\{X(\omega) \mid \omega \in \Omega\}$ is finite / countably infinite

$f(x) = P_X(x) = P(X = x)$

$F(x) = F_X(x) = P(X \leq x)$

$\boxed{f : \mathbb{R} \to [0,1]}$ → $\sum_i f(x_i) = 1$

where $p(X = x_i) \neq 0$

$F : \mathbb{R} \to \mathbb{R}$

Q. what properties of the fn F can make it a PDF for a random variable X!

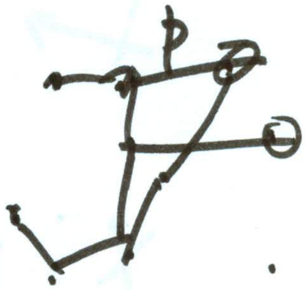
Obs: given f or F,

any $x \in \mathbb{R}$ defines an event.

$$\{X \le x\} = \{\omega \in \Omega \mid X(\omega) \le x\}$$

$$\{X = x\} = \{\omega \in \Omega \mid X(\omega) = x\}$$

$$f(k) = P\{X = k\}$$



$$P(k_1 \le X \le k_2)$$

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wheel of fortune

$$m_1, m_2, \dots, m_n$$

$$p_1, p_2, \dots, p_n$$

$$p_1 m_1 + p_2 m_2 + \dots + p_n m_n$$

can be considered as the expected value in any particular play.

Let K_1 times — m_1

K_2 times — m_2

$\vdots$

K_n $\vdots$ — m_n

$$\text{Expected value} = \frac{K_1 m_1 + K_2 m_2 + \cdots + K_n m_n}{K}$$

$$= \frac{K_1}{K} m_1 + \frac{K_2}{K} m_2 + \cdots + \frac{K_n}{K} m_n$$

$$= p_1 m_1 + p_2 m_2 + \cdots + p_n m_n$$

is the weighted average of the values of X.

Expectation of a random variable.

(mean)

$X \rightarrow$ R.V.

$$\mu = E[x] = E(X) = \sum_x x\, P(X = x)$$

$$= \sum_x x f(x)$$

Example of "no-finite" expectati[on] of a R.V.

1. Consider a random variable X which takes the value 2^k with probability 2^{-k}, $k = 2, 3, \dots$

$$\sum_k 2^k \cdot 2^{-k} = \sum 1$$

2. Consider the density fn. as

$$f(x) = \begin{cases} \frac{1}{x(x+1)}, & k = 1, 2, \dots \\ 0, & \text{otherwise.} \end{cases}$$

$$\sum_{k=1}^{\infty} f(k) = 1 \ ?$$

$$\sum_k f(k) = \sum_k \frac{1}{k(k+1)} = \sum_k \frac{1}{k} - \frac{1}{k+1} = 1$$

$$E(X) = \sum_k k f(k) = \sum_k \frac{1}{k+1} \not\to \text{converge?}$$

$$\sum_n \frac{1}{n^p}$$

Exp. Consider two independent coin tosses, each with a 3/4 probability of a head, and let X be the # of heads obtained.

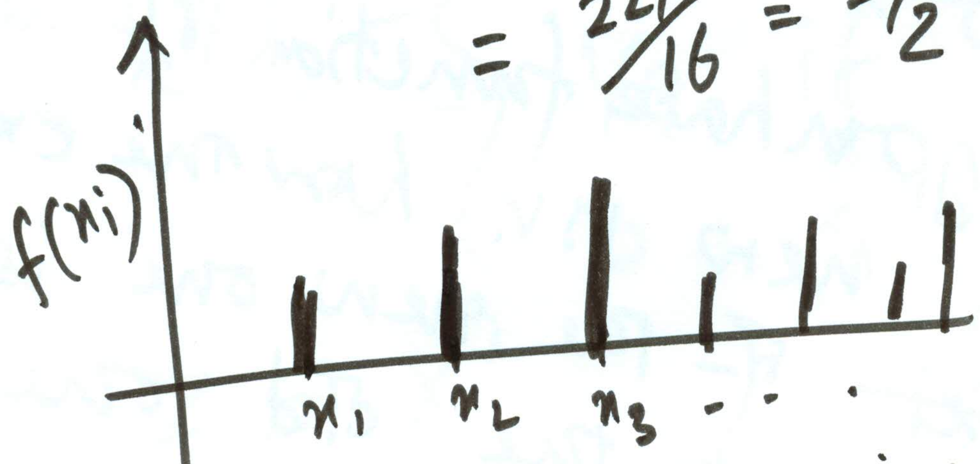
$$E[X] = ?$$

A.

$$= \sum_{k=0,1,2} k f(k)$$

$$= 24/16 = 3/2.$$

PMF



Center of gravity: Given a bar with weight $f(x)$ placed at x with $f(x) \geq 0$, the center of gravity c is the pt. at which the sum of the torques from the weights to its left is equal to the sum of the torques from the weights in its right:

$$\sum (c-x) f(x) = 0 \Rightarrow c = \sum x f(x)$$

let X = today's temprature in degree Celsius

let $Y = 1.8X + 32$, The temprature in degrees Fahrenheit.

$$g(x) = \boxed{Y = aX + b}$$

$\cancel{E(X)}$

If you have function of a r.v. as a new r.v. how the expectation ~~is rel~~ of the new one is related to the old one!!!

The PMF of Y is

$$f(y) = \sum_{\{x \mid g(x) = y\}} f(x)$$

Exp. 1.

let $Y = |X|$

and $f(x) = \begin{cases} 1/9, & x = -4, -3, -2, -1, 0, 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$

$f(y) = \begin{cases} 2/9 & \text{if } y = 1, 2, 3, 4 \\ 1/9 & \text{if } y = 0 \\ 0, & \text{elsewhere} \end{cases}$

let $Z = X^2$

$f(z) = \begin{cases} 2/9, & z = 1, 4, 9, 16 \\ 1/9, & z = 0 \\ 0, & \text{elsewhere.} \end{cases}$

Let $g(X) = Y$

Calculate the PMF of Y using the PMF of X.

$\boxed{NO}$.

$f(\cancel{y})$

But it may help to calculate expectation.

$$X, \quad Y = g(X)$$

$$E[Y] = \sum_y y \hat{f}(y)$$

$$\cancel{= \sum y f_y}$$

$$= \sum_y y \left(\sum_{\{x \mid g(x) = y\}} f(x) \right)$$

$$= \sum_y \sum_{\{x \mid g(x) = y\}} y f(x)$$

$$= \sum_y \sum_{\{x \mid g(x) = y\}} g(x) f(x)$$

$$= \sum_x g(x) f(x) .$$

In order to calculate $E[g(X)]$, no need to calculate ~~f(g(x~~ $\hat{f}(y)$!!

$g(x) = X^n$

$E[x^n] \rightarrow$ n-th "moment" of X (expectation of X)

$= \sum x^n f(x)$

<u>variance.</u>

Let X be a r.v.

$var(X) = E[(X - E[X])^2]$

Since $(X - E[X])^2 \geq 0$

so $var(X) \geq 0$.

Standard deviation

<u>is meaningful.</u> $\sigma = \sqrt{var(X)}$

$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

$$= E[X^2 - 2XE[X] + E[X]^2]$$

$$= E[X^2] - (E[X])^2$$

In order to prove this

Let $Y = aX + b = g(X)$

$$E[Y] = \sum_x (ax+b) f(x)$$

$$= a \sum x f(x) + b \sum_x f(x)$$

$$= a E[X] + b$$

Q. Am I using linearity of E?

$$Y = aX + b$$

$$\text{var}(Y) = E\left[(Y - E(Y))^2\right]$$

due to the formu.

$$= \sum_x \left[(ax + b - E[ax+b])^2 f(x) \right]$$

$$= a^2 \text{var}(X)$$

$$\boxed{\text{var}(X) = E[X^2] - (E[X])^2}$$

proof in HW.

~~Prob.~~ If the weather is good (which happens with prob. 0.6), you walk ~~the~~ 2 kms to class at ~~at~~ a speed of $V = 5$ km/hour, and otherwise you ride your cycle at a speed $V = 30$ km/hr. What is the mean of time T to get to the class? Find out PMF of T

Ans.- ~~Ans.~~

$$f(t) = \begin{cases} 0.6, & t = 2/5 \text{ hrs.} \\ 0.4, & t = 2/30 \text{ hrs.} \end{cases}$$

$$E[T] = \frac{4}{15}$$

Q. Expectation of

= Bernoulli distribution (X)

$$f(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \end{cases}$$

$$E[X] = p$$

$$var[X] = E[X^2] - (E[X])^2$$

$$= p - p^2 = p(1-p)$$

Q. <u>Discrete Uniform RV (X)</u>

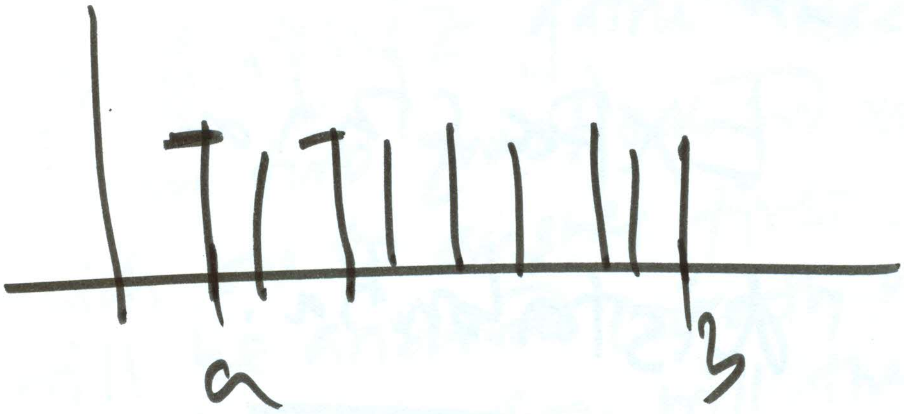
=

$$f(k) = \begin{cases} 1/6, & k = 1,2,\dots,6 \\ 0, & \text{o/w} \end{cases}$$

$$E(X) = 3.5$$

$$var(X) = 35/12$$

x3.



$$f(k) = P(X=k) = \begin{cases} \frac{1}{b-a+1}, & \text{if } k = a, a+1, \dots, b \\ 0 & \text{otherwise.} \end{cases}$$

$E(X) = P. \dfrac{a+b}{2}$ assume $a=1, b=n$

$\mathrm{var}(X) = \dfrac{n^2-1}{12}$

$= E[X^2] - E(X)^2$

$K \cdot W -$

8. Expectation of Binomi[al]
distribution.

$$f(K) = \binom{n}{K} p^K (1-p)^{n-K}, \quad K = 0, 1, \ldots n$$

$$E[X] = \sum_{K=0}^{n} K f(K)$$

$$= \sum K \binom{n}{K} p^K (1-p)^{n-K}$$

$$K \binom{n}{K} = \frac{n(n-1)!}{(K-1)!\,[(n-1)-(K-1)]!} = n\binom{n-1}{K-1}$$

$$E[X] = n \sum_{K=1}^{n} \binom{n-1}{K-1} p^K (1-p)^{n-K}$$

let $i = K-1$

$$E[X] = np \underbrace{\sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-i-1}}_{1}$$

$$= np$$

Exp. Consider a quiz game where a person is given two questions and must decide which one to answer first. Question 1 will be answered correctly with prob. 0.8 and the first person will then receive as prize ₹ 100, while question 2 will be answered correctly with prob. 0.5 and the person then receive an prize ₹ 200.

If the first question attempted is answered incorrectly, the quiz terminates i.e. the person is not allowed to attemp the 2nd question.

If the first question is answered correctly, the person is allowed to attempt the 2nd question.

Which question should be answered first to maximize the expected value of the total prize money received!