Lecture-5

Probability & Statistics.

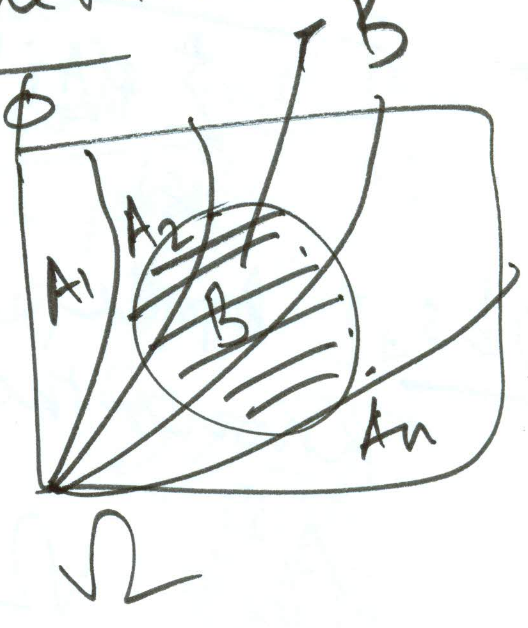
Conditional probability

$P(A \mid B)$

and its applications.

Total probability theorem.

$\bigcup_{i=1}^{n} A_i = \Omega$ $A_1, \ldots, A_n$, $A_i \cap A_j = \phi$



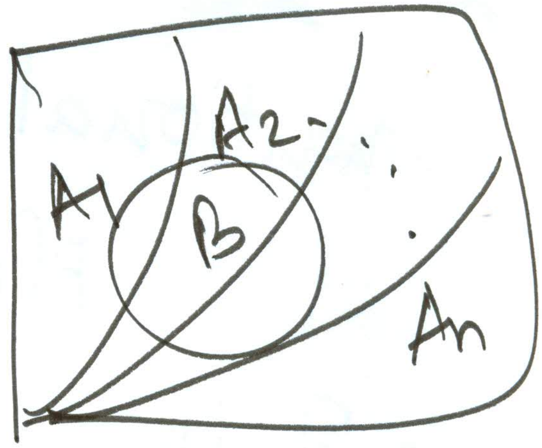
$$P(B) = P\left(\bigcup_i \underline{A_i \cap B}\right)$$

$$= \sum_i P(A_i \cap B)$$

$$= \sum_i \underbrace{P(A_i)}_{\text{Known}} \underbrace{P(B \mid A_i)}_{\text{Known}}$$

Baye's theorem.

$P(A_i|B) \quad P(B|A_i) . \quad P(B) > 0$



Ω

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)}$$

$$= \frac{P(A_i)\; P(B|A_i)}{\sum_{i=1}^{n} P(A_i)\; P(B|A_i)}$$

Obs. Apply BT / TPT when you know the probability of $A_i \subseteq \Omega$

s.t. $\Omega = \bigcup_{i=1}^{n} A_i$

and $A_i \cap A_j = \phi \;,\; i \neq j$

prob.

given

A test for a certain rare disease is assumed to be correct 95% of the time: If the person has the disease, the results are positive with prob. 0.95, and if the person does not have the disease, the test results are negative with prob. 0.95.

Assumption.

An random person drawn from a certain population has prob. 0.001 of having the disease.

Q.

Given that the person just tested positive, what is the prob. of having the disease?

$B \equiv$

$A =$ ~~test result is positive~~ $P(B|A)$

$P(\cancel{A|B})$ $B|A$

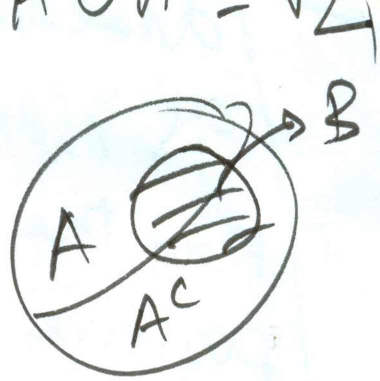
$P(A|B)$

$A \rightarrow$ the event that a person has the disease.

$B \rightarrow$ the event that the test result are positive

$P(B|A) = 0.95$. | $A \cup A^c = \Omega$.

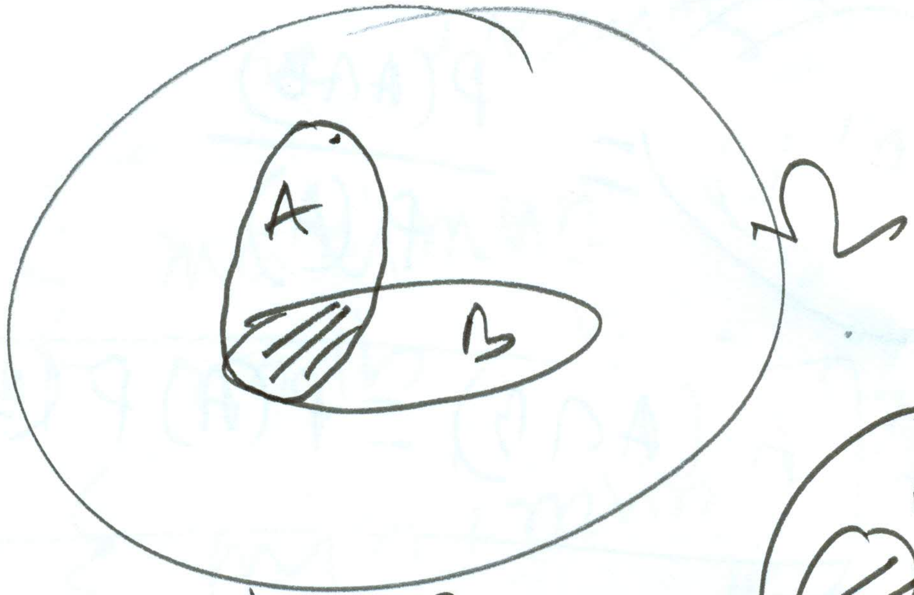
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



$$= \frac{P(A)\,P(B|A)}{P(A)\,P(B|A) + P(A^c)\,P(B|A^c)}$$

$$= \frac{0.001 \times 0.95}{(0.001 \times 0.95) + (0.999 \times 0.05)}$$

$$= 0.0187$$



$P(A|B) = ?$

$P(B|A) = ?$

$P(A) > 0$

$P(B) > 0$

$$\boxed{P(A|B) = P(A)}$$

$$\frac{P(A\cap B)}{P(B)} = P(A)$$

$$\Rightarrow \boxed{P(A\cap B) = P(A)P(B)}$$

A & B are independent events

0 = nonzero

$$P(B) = P(B|A) = \frac{P(A\cap B)}{P(A)}$$

$$\Rightarrow \boxed{A(A\cap B) = P(A)P(B)}$$

Obs. Independence is a symmetric relation:

A is independent of B

$\iff$ B is independent of A.

Prob. 1 Consider an experiment involving two successive rolls of a 4-sided die in which all 16 outcomes are equally likely and have prob. 1/16.

(A) Are the events

$A_i = \{$1st roll results in $i\}$

$B_j = \{$2nd roll results in $j\}$

independent?

(B) Are the events

$A = \{$ maximum of two rolls is 2 $\}$

$B = \{$ minimum of two rolls is 2 $\}$

$n \to$ # of persons.

Q. What is the max. # of edges in this network?

$^nC_2$

$^nC_2 + {}^nC_3 + {}^nC_4$

$A_1, A_2, \dots, A_n$ events of Ω.

are independent if

$$P\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} P(A_i)$$

where S is any subset of $\{1, 2, \dots, n\}$

Q. $n = 3$

A_1, A_2, A_3

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$
$$P(A_1 \cap A_3) = P(A_1)P(A_3)$$
$$P(A_2 \cap A_3) = P(A_2)P(A_3)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

Exm. Consider two independent fair coin tosses, and the following events

$H_1 = \{ 1^{st} \text{ toss is a head} \}$

$H_2 = \{ 2^{nd} \text{ toss is a head} \}$

$D = \{ \text{the two tosses have different results} \}$

$$P(H_1 \cap H_2) = P(H_1)\,P(H_2)$$

$$P(H_1 \cap D) = P(H_1)\,P(D)$$

$$P(H_1 \mid D) = P(H_1) \quad P(D \mid H_1) = P(D)$$

Prob. Consider two independent rolls of a fair six-sided die, and the following events

$A = \{ 1^{st} \text{ roll is } 1, 2, \text{ or } 3 \}$

$B = \{ 1^{st} \text{ roll is } 3, 4 \text{ or } 5 \}$

$C = \{ \text{the sum of the two rolls is } 9 \}$

$$P(A \cap B) = \frac{1}{6} \neq \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(B)$$

$$P(A \cap B \cap C) = \frac{1}{36} = \underset{1/2}{P(A)}\, \underset{1/2}{P(B)}\, \underset{4/36}{P(C)}$$

$$P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$$

$$P(A \cap B) = P(A)\, P(B)$$

$$P(A \cap B \mid \Omega) = P(A \mid \Omega)\, P(B \mid \Omega)$$