Lecture-18

Probability & Statistics.

Moment generating function.
(MGF)

$$\psi(t) = E[e^{tX}].$$

Markov's Inequality.

$$P(X \geq a) \leq \frac{E[X]}{a}$$

$X \geq 0$
$a > 0$

Chebyshev's inequality.

$E[X] = \mu$

$$P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}, \quad a > 0.$$

$var(X) = \sigma^2$

Prob. Suppose that it is known that the # of items produced in a factory during a week is a random variable with mean 50.

(a) What can be said about the probability that this week's production will exceed 75?

(b) If the variance of a week's production is known to equal 25, then what can be said about the probability that this week's production will be between 40 & 60?

Ans. (a) $2/3$ (b)

-(5)

$$P\{|X-50| \geq 10\} \leq \frac{\sigma^2}{a^2} = \frac{25}{10^2} = \frac{1}{4}$$

$$P\{|X-50| \leq 10\}$$

$$= P\{40 \leq X \leq 60\}$$

$$\geq 1 - \frac{1}{4} = \frac{3}{4}.$$

Law of Large numbers.

Weak law of large numbers

It states that the probability that the average of the first n terms in a seq: of independent & identically distributed random variable differs by its mean by more than ε goes to 0 as $n \to \infty$.

$$P\left\{\left|\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right| > \varepsilon\right\} \to 0 \text{ as } n \to \infty$$

$$E[X_i] = \mu, \quad \operatorname{var}(X_i) = \sigma^2$$

pf:

$$E\left[\frac{X_1+X_2+\cdots+X_n}{n}\right]=\mu$$

$$\operatorname{Var}\left[\frac{X_1+X_2+\cdots+X_n}{n}\right]=\frac{\sigma^2}{n}$$

$$P\left\{\left|\frac{X_1+X_2+\cdots+X_n}{n}-\mu\right|>\varepsilon\right\}\leq\frac{\sigma^2}{n\varepsilon^2}$$

by Chebyshev's inequality. as $n\to\infty$ $\to 0$ get.

Strong law of large numbers.

Thm. Let $X_1, X_2, \cdots, X_n$ be a seqr of independent random variables having a common distribution, and let $E[X_i]=\mu$. Then with prob. 1

$$\frac{X_1+X_2+\cdots+X_n}{n}\to\mu \text{ as } n\to\infty.$$

As an example, suppose a seq. of independent trials is performed. Let E be a fixed event and denote P(E) the prob. that E occurs on any particular trial.

Let $X_i = \begin{cases} 1 & \text{if } E \text{ occurs on the } i\text{th trial} \\ 0 & \text{if } E \text{ does not occur on the } i\text{th trial.} \end{cases}$

By strong law of large numbers

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \to E[X_i] = P(E)$$

when n is large.

Central Limit Theorem.

↓

the number one theorem in probability.

any

Let $X_1, X_2, \ldots$ be a seq. of independent, identically distributed random variables, each with mean μ and variance σ^2.

Then the distribution of

$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$.

$$P\left\{ \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le a \right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2}\, dx$$

$$\|$$

Ex. 1

Let X_i, $i = 1, 2, \ldots, 10$ be independent random variables, each being uniformly distributed over $(0, 1)$.

Estimate $P\left\{ \sum_{i=1}^{10} X_i > 7 \right\}$,

Sol.

$$\mu = \frac{1}{2}, \quad \sigma^2 = \frac{1}{12}$$

$$P\left\{ \sum_{i=1}^{10} X_i > 7 \right\}$$

$$P\left\{ \frac{\sum_{i=1}^{10} X_i - 5}{\sqrt{10 \cdot 1/12}} > \frac{7-5}{\sqrt{10 \cdot 1/12}} \right\}$$

$$P\left\{ \downarrow \quad > \quad \right\}$$

$$1 - \Phi(2.2)$$

$$= 0.0139.$$