~~Lecture-16.~~

This is Lecture 17

Probability & Statistics.

$\boxed{\text{Conditional distribution.}}$

can be thought of as a technique to understand the co-relationship between two random variables.

$$f(x,y) \longrightarrow f_Y(y).$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

Function of two random variables.

let $Y = H(X_1, X_2)$ where X_1 & X_2 are random variables.

How to determine the density function of Y.

Algo: (1) Let $Y = H_1(X_1, X_2)$

(2) Introduce a second random variable $Z = H_2(X_1, X_2)$. The fn H_2 is selected for convenience, but we want to be able to solve $y = H_1(x_1, x_2)$ and $z = H_2(x_1, x_2)$ for x_1, x_2 in terms of y & z.

③ Find
$$x_1 = a_1(y,z)$$
$$x_2 = a_2(y,z)$$

④ Find the partial derivatives (we assume that they exist and continuous)

$$\frac{\partial x_1}{\partial y}, \quad \frac{\partial x_1}{\partial z}, \quad \frac{\partial x_2}{\partial y}, \quad \frac{\partial x_2}{\partial z}$$

⑤ The joint density fn. of (Y, Z) denoted by $l(y,z)$ is found as

int
nsity fn.

$$l(y,z) = f\left(a_1(y,z), a_2(y,z)\right) \left| J(y,z) \right|$$

where
$$J(y,z) = \begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{vmatrix}$$

(Jacobian)

The density of Y say g_Y is then found as

$$g_Y(y) = \int_{-\infty}^{\infty} l(y,z)\, dz .$$

Exp 1: Let X_1 & X_2 be two random variables with joint density fn. as

$$f(x_1, x_2) = \begin{cases} 4e^{-2(x_1+x_2)}, & x_1 > 0,\ x_2 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

~~Suppose we are~~, calculate the density fn. of $Y = \frac{X_1}{X_2}$.

Soln. If we choose $z = x_1 + x_2$

~~then~~, $x_1 = \frac{yz}{1+y}$, $x_2 = \frac{z}{1+y}$

then $\frac{\partial x_1}{\partial y} = \frac{z}{(1+y)^2}$, $\frac{\partial x_1}{\partial z} = \frac{y}{1+y}$

$\frac{\partial x_1}{\partial y} = -\frac{z}{(1+y)^2}$, $\frac{\partial x_2}{\partial z} = \frac{1}{1+y}$

$\therefore J(y,z) = \left| \begin{matrix} \cdot & \cdot \\ \cdot & \cdot \end{matrix} \right| = \frac{z}{(1+y)^2}$

and $f(h_1(y,z), h_2(y,z)) = 4e^{-2z}$.

$\therefore \ell(y,z) = 4e^{-2z} \cdot \frac{z}{(1+y)^2}$ and

$g_Y(y) = \int_0^\infty 4e^{-2z} \frac{z}{(1+y)^2} dz = \begin{cases} \frac{1}{(1+y)^2}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$

Moment Generations function,

and its application.

Q. If X_1, X_2 and i.i.d random variable then does $X_1 + X_2$ have similar/same density fn?

The ~~moment generating fn~~ $\phi(t)$ is of the random variable X defined for all values of t by

$$\phi(t) = E[e^{tX}].$$

$$= \begin{cases} \sum_x e^{tx} f(x), & \text{discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\, dx & \text{if } X \text{ is contiuous.} \end{cases}$$

This is called 'mgf' because all the moments of X can be obtained by successively differentiating $\varphi(t)$. For example,

$$\phi'(t) = \frac{d}{dt}\left(E[e^{tX}]\right)$$
$$= E\left[\frac{d}{dt} e^{tX}\right]$$
$$= E\left[X e^{tX}\right]$$

Then $\phi'(0) = E[X]$

Similarly, $\phi''(t) = E\left[\frac{d}{dt}(Xe^{tX})\right]$

$$= E\left[X^2 e^{tX}\right]$$

Then $\phi''(0) = E[X^2]$

$\vdots$

In general $\phi^{(n)}(0) = E[X^n]$

An interesting property of MGF.

Let X & Y be independent.

$$\phi_{X+Y}(t) = E\left[e^{t(X+Y)}\right]$$

$$= E\left[e^{tX} e^{tY}\right]$$

(recall $E[XY] = E[X]\,E[Y]$ if X & Y are independent)

$$= E\left[e^{tX}\right] E\left[e^{tY}\right]$$

$$= \phi_X(t)\ \phi_Y(t)$$

$$\boxed{\phi_{X+Y}(t) = \phi_X(t)\ \phi_Y(t)}$$

important.

<u>Exp</u>. Compute the MGF of the Poisson Random variable:

$$P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k=0,1,2,\ldots$$

$$\begin{aligned}
\phi(t) &= E[e^{tX}] \\
&= \sum_{k=0}^{\infty} e^{tk} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k} e^{tk} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} e^{\lambda e^t} \\
&= \exp\{\lambda(e^t - 1)\}.
\end{aligned}$$

Then $E[X] = \phi'(0) = \lambda$

$$\begin{aligned}
\mathrm{var}(X) &= \phi''(0) - (E[X])^2 \\
&= \lambda^2 + \lambda - \lambda^2 = \lambda
\end{aligned}$$

Exp. Exponential random ~~var~~ variable.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then $\phi(t) = E[e^{tX}]$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x} \, dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x} \, dx$$

$$= \frac{\lambda}{\lambda - t}, \text{ if } t < \lambda.$$

Then $E[X] = \phi'(0) = 1/\lambda$

$$\mathrm{var}(X) = \phi''(0) - (E[X])^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Exp 3. Gamma distribution.

(α, λ), $\lambda > 0, \alpha > 0$.

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\phi(t) = E[e^{tX}]$$
$$= \int_0^\infty e^{tx} f(x)\, dx$$
$$= \left(\frac{\lambda}{\lambda - t}\right)^\alpha \underbrace{\frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} y^{\alpha-1}\, dy}_{=1} \qquad y = (\lambda - t)x$$
$$= \left(\frac{\lambda}{\lambda - t}\right)^\alpha.$$

$$E[X] = \phi'(0) = \frac{\alpha}{\lambda}, \quad Var(X) = \phi''(0) - \left(\frac{\alpha}{\lambda}\right)^2$$
$$= \alpha / \lambda^2$$

One important property of MGF:

it uniquely specifies the distribution,

for example,

(1) Let X_1 & X_2 be independent Poisson random variables having respective means λ_1 & λ_2.

$$E[e^{t(X_1+X_2)}] = E[e^{tX_1}]\,E[e^{tX_2}]$$

$$= \exp\{\lambda_1(e^t-1)\}\exp\{\lambda_2(e^t-1)\}$$

$$= \exp\{(\lambda_1+\lambda_2)(e^t-1)\}$$

Because $\exp\{(\lambda_1+\lambda_2)(e^t-1)\}$ is the MGF of a Poisson r.v. having mean $\lambda_1+\lambda_2$, we conclude from the fact that the MGF uniquely specifies the distribution, X_1+X_2 is Poisson with mean $\lambda_1+\lambda_2$.

Ex2. Let X_1 & X_2 be independent random variables with parameters (α_1, λ) & (α_2, λ) (Gamma distribution) then $X_1 + X_2$ is a Gamma random variable with parameter $(\alpha_1 + \alpha_2, \lambda)$.

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\,\phi_{X_2}(t)$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1} \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_2}$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1 + \alpha_2}$$

which is seen to be the MGF of a gamma $(\alpha_1 + \alpha_2, \lambda)$ random variable.

Prop. If X_i, $i = 1, \ldots, n$ are independent gamma random variables with respective parameters (α_i, λ) then $\sum_{i=1}^{n} X_i$ is gamma with parameter $\left(\sum_{i=1}^{n} \alpha_i, \lambda\right)$.

Since $\alpha = 1$ in Gamma gives the exponential distribution with rate λ, we have the following:

If $X_1, X_2, \ldots, X_m$ are independent exponential random variables each having rate λ, then

$$\sum_{i=1}^{m} X_i$$ is a Gamma random variable with parameters (m, λ).

Markov inequality.

If X is a random variable that takes only nonnegative values, then for any value $a > 0$

$$P\{X \geq a\} \leq \frac{E[X]}{a}.$$

Pf. $E[X] = \int_0^\infty x f(x)\, dx$

$$= \int_0^a x f(x) + \int_a^\infty x f(x)$$

$$\geq \int_a^\infty x f(x)$$

$$\geq \int_a^\infty a f(x)$$

$$= a \int_a^\infty f(x)\, dx$$

$$= a P(X \geq a)$$

A corollary is the Chebyshev's inequality.

If X is a random variable with mean μ and variance σ^2, then for any value $K > 0$

$$P\{|X-\mu| \geq K\} \leq \frac{\sigma^2}{K^2}$$

Pf. Since $(X-\mu)^2$ is nonnegative r.v., we can apply Markov's inequality with $a = K^2$ to obtain

$$P\{(X-\mu)^2 \geq K^2\} \leq \frac{E[(X-\mu)^2]}{K^2} \quad -(1)$$

But since $(X-\mu)^2 \geq K^2$ iff $|X-\mu| \geq K$, the eqn (1) is equivalent to

$$P(|X-\mu| \geq K) \leq \frac{E[(X-\mu)^2]}{K^2} = \frac{\sigma^2}{K^2}$$