This is Lecture 16

~~Lecture 15~~.

Probability & Statistics.

Jointly distributed random variables.

marginal density f_X, f_Y

$$f(x,y) = f_X(x)\, f_Y(y)$$

$\Leftrightarrow$ X & Y are independent.

Q. why to study covariance?

Ans. It can give an 'idea' about closeness of two random variables.

[recall the properties of scalar product (inner product) and match it with the properties of covariance (consider it as a function), and "orthogonality $\approx$ independent"

~~Given two random variables~~

~~X & Y.~~

Consider two indicator random variables:

$$X = \begin{cases} 1 & \text{if the event } A \text{ occurs} \\ 0 & \text{if } A \text{ does not occur.} \end{cases}$$

$$Y = \begin{cases} 1 & \text{if the event } B \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

$$XY = \begin{cases} 1 & \text{if both } A \text{ \& } B \text{ occur} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{Cov}[X\,Y] = E[XY] - E[X]\,E[Y]$$
$$= P\{X=1, Y=1\} - P\{X=1\}\,P\{Y=1\}$$

If $Cov(X,Y) > 0$

$$P\{X=1, \cancel{X}=1\} > P\{X=1\} P\{Y=1\}$$

$$\Rightarrow \frac{P\{X=1, Y=1\}}{P\{Y=1\}} > P\{X=1\}$$

$$\Rightarrow P(X=1 \mid Y=1) > P(X=1) \quad -(1)$$

Illy, if $cov(X,Y) < 0$

$$\Rightarrow P(X=1 \mid Y=1) < P(X=1) \quad -(2)$$

From (1) & (2) we can say.

$Cov(X,Y) > 0 \Rightarrow Y \uparrow$ if $X \uparrow$

$Cov(X,Y) < 0 \Rightarrow Y \downarrow$ if $X \uparrow$

Thus, the strength of the relationship between X & Y is indicated by the correlation between X & Y, a dimensionless quantity obtained by dividing the covariance by the product of the standard deviation of X & Y

$$\text{Thus Corr Coef}(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)\, Var(Y)}}$$

$$= \rho \rightarrow \text{notation.}$$

H.W. $\cancel{Cov}\ -1 \leq \rho \leq 1$

If $\rho = 1 \rightarrow$ highly +vely correlated.

If $\rho = -1 \rightarrow$ highly -vely correlated.

Conditional distribution.

The relationship between two r.v. can often be clarified by considerin of the conditional distribution of one given the value of other one.

Recall the defn. of conditional prob.

$$P(E|F) = \frac{P(E \cap F)}{P(F)}, \quad P(F) > 0$$

Hence if X & Y are discrete random variables, it is natural to define the conditional PMF of X given $Y = y$ by

$$p_{X|Y}(x|y) = P\{X = x \mid Y = y\}$$

$$= \frac{P\{X = x, Y = y\}}{P\{Y = y\}}$$

$$= \frac{f(x,y)}{f_Y(y)}.$$

Prob! Suppose that $f(x,y)$, the joint PMF of X & Y is given by

$f(0,0) = 0.4$, $f(0,1) = 0.2$,
$f(1,0) = 0.1$, $f(1,1) = 0.3$.

Calculate the conditional PMF of X given that $Y = 1$.

Sol: $f_{X|Y}(x|1)$. $\quad y = 1$

$$= \frac{f(x,y)}{f_Y(y)}$$

$$f_Y(y) = f_Y(1) = \sum_x f(x,1) = 0.5$$

Hence, $P\{X=0 \mid Y=1\} = \dfrac{f(0,1)}{P(Y=1)} = \dfrac{f(0,1)}{f_Y(1)} = 2/5$

$P\{X=1 \mid Y=1\} = \dfrac{f(1,1)}{f_Y(1)} = 3/5$

If X & Y are jointly continuous.

Then

$$\cancel{P\{X} \quad f_{X|Y} = \frac{f(x,y)}{f_Y(y)}.$$

Exp. The joint density of X & Y is given by

$$f(x,y) = \begin{cases} \frac{12}{5}\, x(2-x-y), & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise.} \end{cases}$$

Compute the conditional density of X, given that $Y=y$, where $0<y<1$.

Ans. $$f_{X|Y}(x|y) = \frac{6x(2-x-y)}{4-3y}.$$