~~Lecture 14~~

This is Lecture 15

Probability & Statistics.

Joint probability mass function.

↳ at least two random variables are involved

$$p(X = x_i, Y = y_j) = \text{given}$$

$$p(X = x_i) = \sum_j p(x_i, y_j)$$

We say that X & Y are jointly continuous if $\exists$ a fn. $f(x,y)$ defined for all real x and y having the property that for every subset $C \subseteq \mathbb{R}^2$

$$P\{(X,Y) \in C\} = \iint_C f(x,y)\,dx\,dy.$$

$$P(X \in A, Y \in B) = \int_B \int_A f(x,y)\, dx\, dy$$

$$F(a,b) = P(X \leq a, Y \leq b)$$

$$= P\{X \in (-\infty, a], Y \in (-\infty, b]\}$$

$$= \int_{-\infty}^{b} \int_{-\infty}^{a} f(x,y)\, dx\, dy$$

Then $f(a,b) = \dfrac{\partial^2}{\partial a \partial b} F(a,b)$.

where partial derivatives are defined.

density.

~~more~~ interpretation,

$$P(a < X < a + da,\ b < Y < b + db)$$

$$= \int_b^{b+db} \int_a^{a+da} f(x,y)\, dx\, dy$$

$$\approx f(a,b)\, da\, db$$

Hence $f(a,b)$ is a measure how likely it is that the random ~~var~~ vector (X,Y) will be near (a,b).

If X & Y are jointly continuous, they are individually continuous and their PDF can be written.

$$P(X \in A) = P(X \in A, -\infty < Y < \infty)$$

$$= \int_A \left(\int_{-\infty}^{\infty} f(x,y)\, dy \right) dx.$$

$$= \int_A f_X(x)\, dx$$

where $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx .$$

Prob. The joint pdf of X & Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & 0<x<\infty,\ 0<y<\infty \\ 0, & \text{otherwise} \end{cases}$$

Compute:
a) $P(X>1, Y<1)$
b) $P(X<Y)$
c) $P(X<a)$.

Ans. a) $e^{-1}(1-e^{-2})$

b) $\frac{1}{3}$

$$P(X<Y) = \iint_{x<y} 2e^{-x}e^{-2y}\, dx\, dy$$
$$= \int_0^{\infty}\left(\int_0^{y} 2e^{-x}e^{-2y}\, dx\right) dy$$

c) $1-e^{-a}$

Independent random variables.

The r.v.s X & Y are said to be independent if for any two sets of real numbers A & B.

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$$

otherwise. The events

$$E_A = \{X \in A\} \;\&\; E_B = \{Y \in B\}$$

are independent.

Hence. $\boxed{F(a,b) = F_X(a)\, F_Y(b)}$

<u>Them</u>. Let X & Y be ~~two independent~~ discrete random variables. Then X & Y are independent iff

$$f(x,y) = f_X(x)\, f_Y(b)$$

where $f_X(x)$ & $f_Y(x)$ are the individual PMFs.

(6)

'$\Rightarrow$' part.

Set $A = \{x\}$, $B = \{y\}$.

'$\Leftarrow$' part.

$$P(X \in A, Y \in B)$$

$$= \sum_{y \in B} \sum_{x \in A} f(x,y) \xrightarrow{\text{by assumption}}$$

$$= \sum_{y \in B} \sum_{x \in A} f_X(x) f_Y(y)$$

$$= \sum_{x \in A} f_X(x) \sum_{y \in B} f_Y(y)$$

$$= P(X \in A) \; P(Y \in B)$$

Note: Loosely speaking, X & Y are independent if knowledge of the value of one does not change the distribution of the other.

Exp. Suppose X & Y are independent random variables having the common density function (CDF)

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the density fn of the r.v. X/Y.

$$F_{X/Y}(a) = P(X/Y \le a) = \iint\limits_{x/y \le a} f(x,y)\, dx\, dy$$

$$= 1 - \frac{1}{a+1}$$

Now differentiate $F_{X/Y}(a)$ to get the density fn of X/Y which is

$$f_{X/Y}(a) = \frac{d}{da} F_{X/a}(a) = \frac{1}{(a+1)^2},$$

$$0 < a < \infty.$$

We can extend the 'idea' for any 'n' numbers of random variables.

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$$

$$E(X+Y) = E(X) + E(Y)$$

$$\mathrm{var}(X+X) \neq 2\,\mathrm{var}(X)$$

$$\mathrm{var}(aX+b) = a^2\,\mathrm{var}\,X$$

Let us introduce the concept of covariance of two random variables.

Defⁿ. $\mathrm{Cov}(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$

mean of X mean of Y

$$= E[XY] - E[X]\,E[Y]$$

Properties of covariance as a fn.

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(1) $Cov(X,Y) = Cov(Y,X)$

storing lont important.

(2) $Cov(X,X) = var(X)$

(3) $Cov(aX,Y) = a\,Cov(X,Y)$.

(4) $Cov(X_1+X_2, Y) = Cov(X_1,Y) + Cov(X_2,Y)$

(5) $Cov\left(\sum_{i=1}^{n} X_i, Y\right) = \sum_{i=1}^{n} Cov(X_i,Y)$

(6) $Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{M} Cov(X_i,Y_j)$

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$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right)$$

$$= \sum_{i=1}^{n} \mathrm{var}(X_i) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} \mathrm{Cov}(X_i, X_j)$$

For $n = 2$,

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + \mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X)$$

$$= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$$

Thm. If X & Y are independent r.v. then $\mathrm{Cov}(X,Y) = 0$.

and so for independent $X_1, \dots, X_n$

$$\mathrm{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{var}(X_i).$$

Exp. Compute the variance of the sum obtained when 10 independent rolls of a fair die are made.

Sol^n. Let X_i denote the outcome of the ith roll.

Then we have

$$var\left(\sum_{i=1}^{10} X_i\right) = \sum_{i=1}^{10} var(X_i).$$

$$var(X_i) = E[X_i^2] - (E[X_i])^2$$

$$E[X_i^2] = \sum_{i=1}^{6} i^2 P(X=i) = \frac{91}{6}$$

$$E[X] = \sum i P(X=i) = 7/2$$

$$var(X_i) = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

$$\therefore var\left(\sum_{i=1}^{10} X_i\right) = \frac{175}{6}.$$