Probability & Statistics

L8

Discrete Random Variables.

$$(\Omega, \mathcal{A}, P).$$

Ω countable, $\mathcal{A} = \mathcal{P}(\Omega)$, P probability measure.

$$X : \Omega \to \mathbb{R}$$

Expected value of a random variable.

$$E(X) = \sum_{j=1}^{\infty} x_j \, p(X = x_j)$$

$x_j = X(j) = X(\omega_j)$, $\omega_j \in \Omega$

~~nothing to d.~~ not much to do with ~~but~~ outcomes but the values of the outcomes and the prob. of obtaining that value.!!

Random Vectors.

Suppose we have 'n' random variables defined on a 'fixed' sample space.

$$X_1, X_2, \ldots, X_n \quad \text{RVs.}$$

$$X_j : \Omega \to \mathbb{R} \quad \forall j = 1:n$$

$$\vec{X} = (X_1, X_2, \ldots, X_n)$$

$$\vec{X} : \Omega \to \mathbb{R}^n$$

$$\omega \in \Omega, \ \vec{X}(\omega) = (X_1(\omega), X_2(\omega), \ldots, X_n(\omega))$$

Borel ~~set~~ : σ-field : the σ-field generated by $\prod_{j=1}^{n} [a_j, b_j]$,

$$= [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

$\vec{X}$ is a RV if $\vec{X}^{-1}(B) \in \mathcal{A}$.

Ex.

X_1 = the height

X_2 = the weight

$[a_1, b_1] \rightarrow a_1 = 5$ ft
$b_1 = 5.5$ ft.

$[a_2, b_2] \rightarrow a_2 = 60$ kg.
$b_2 = 80$ kg.

$\vec{X} = (X_1, X_2)$, $B = [a_1, b_1) \times [a_2, b_2]$

$$\vec{X}^{-1}(B) = \{w \in \Omega \mid X_1 \in [a_1, b_1], X_2 \in [a_2, b_2]\}$$

$$= \{w \in \Omega \mid a_1 \leq X_1(w) \leq b_1, a_2 \leq X_2(w) \leq b_2\}.$$

Exp. Roll a die two times. Let X_1 be the max value, X_2 the min value and X_3 the sum of both.

$$\Omega = \{1, 2, \ldots, 6\}^2 = \{1, \ldots 6\} \times \{1, 2 \ldots 6\}$$

Then the 3-dimensional random vector $\vec{X} = (X_1, X_2, X_3)$ maps Ω to $\mathbb{R}^3$.

For example, if $\underset{\overset{\|}{\omega}}{(2,5)} \in \Omega$

$$\vec{X}(\omega)$$

$$= \vec{X}((2,5))$$

$$= (X_1(2,5),\ X_2(2,5),\ X_3(2,5))$$

$$= (5, 2, 7)$$

__Exp.__ Roll number is a random vector.

Assuming that $\vec{X}$ a RV, define the probability distribution of $\vec{X}$

$P_X(B)$ ← for one random value
$= P\{w \in \Omega \mid X(w) \in B\}$.

$B = B_1 \times B_2 \times \cdots \times B_n$

$B_j = [a_j, b_j]$

$$P_{\vec{X}} = P_{(X_1, X_2, .., X_n)}(B),$$

$= P\{w \in \Omega \mid X_1 \in B_1, X_2 \in B_2, .., X_n \in B_n\}$

$= P\{w \in \Omega \mid a_1 \leq \underline{X_1(w) \leq} b_1, \ldots, a_n \leq X_n(w) \leq b_n\}$.

Exp. Roll a die 3 times. Let X_1, X_2 & X_3 be the observed values in the 1st, 2nd & 3rd roll.

If $B = [1,2] \times [0,1] \times [3,4]$

then

$$P_{\vec{X}}(B) = P(\vec{X} \in B)$$
$$= P(X_1 \in [1,2], X_2 \in [0,1], X_3 \in [3,4])$$

$$= \frac{1}{?}$$

Joint distribution.

Let us consider $n = 2$.

We have 2 RVs, X & Y.

$X, Y : \Omega \to \mathbb{R}$,

$X : \Omega \to D = \{x_1, x_2, \ldots\}$

$Y : \Omega \to E = \{y_1, y_2, \ldots\}$

$$(X, Y) \in \{ (x_i, y_j) \mid x_i \in D, \; y_j \in E \}$$

$$(X, Y) \in D \times E.$$

$$P_{(X,Y)} = P\big((X,Y) = (x_i, y_j) \big)$$
$$= P(X = x_i, \; Y = y_j)$$
$$i, j = 1, 2, \ldots$$

$$p_{ij} = P(X = x_i, Y = y_j)$$

$Y/X \rightarrow$	x_1	x_2	$\ldots$
y_1	p_{11}	p_{21}	$\ldots$
y_2	p_{12}	p_{22}	
$\vdots$	$\vdots$	$\vdots$	

<u>Marginal distribution</u>: we are interested about distribution of X and/or Y, given $P_{(X,Y)}$.

Q. Does the joint distribution determine the marginal distribution or/and the joint distribution be defined ~~by~~ from the marginal one?

Given $B \subseteq D \times E$

$$\cancel{P_{\vec{X}}(B) = 1}$$

$$P_{\vec{X}}(B) = \sum_{(i,j):(x_i,y_j)\in B} p_{ij}$$

	x_1	x_2	$\cdots$	
y_1	p_{11}	p_{21}	$\cdots$	r_1
y_2	p_{12}	p_{22}	$\cdots$	r_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$
	q_1	q_2	$\cdots$	1

$$P(X = x_i)$$

$$= P(X = x_i, Y \in E)$$

$$= P(X = x_i, y \in \cup\{y_j\})$$

$$= \sum_{j=1}^{\infty} P(X = x_i, Y = y_j)$$

$$= \sum_{j=1}^{\infty} p_{ij} = q_i$$

/ similarly $r_j = \sum_{i=1}^{\infty} p_{ij}$

A. There are four balls in an urn, two labelled with '0' and another two labelled with 1.

Choose two balls without replacing the first one.

Let X = the value of the first ball

Y = " " " " 2nd ball.

$$P(X=0, Y=0) = \frac{2}{4} \times \frac{1}{3} = \frac{1}{6}$$

$$P(X=1, Y=0) = \frac{2}{4} \times \frac{2}{3} = 1/3$$

$$P(X=0, Y=1) = 1/3$$

$$P(X=1, Y=1) = 1/6 .$$

Y/X	0	1	
0	1/6	1/3	1/2
1	1/3	1/6	1/2
	1/2	1/2	1

Suppose we relace the first ball. Let us denote the values of the 1st & 2nd ball by X' & Y' resp.

$$P(X'=0, Y'=0) = \frac{1}{4}$$

$$P(X'=1, Y'=0) = 1/4$$

$$P(X'=0, Y'=1) = 1/4$$

$$P(X'=1, Y'=1) = 1/4$$

Y'/X'	0	1	
0	1/4	1/4	1/2
1	1/4	1/4	1/2
	1/2	1/2	1

<u>Conclusion</u>: the marginal distributions do not determine joint distribution.

Q. when are n given random variables independent.

$$\vec{X} = (X_1, X_2, \ldots X_n)$$

$$P_{\vec{X}}(B) \qquad B = [a_1, b_1] \times [a_2, b_2] \ldots \times [a_n, b_n]$$

$$= P(\vec{X} \in B)$$

$$= P(X_1 \in [a_1, b_1], X_2 \in [a_2, b_2], \ldots X_n \in [a_n, b_n])$$

$$= P(X_1 \in [a_1, b_1]) \, P(X_2 \in [a_2, b_2]) \cdots P(X_n \in [a_n, b_n])$$

$$= P_{X_1}([a_1, b_1]) \, P_{X_2}([a_2, b_2]) \ldots P_{X_n}([a_n, b_n])$$

For discrete random variables, X & Y are independent

iff $p_{ij} = q_i \, r_j$.

Mapping of random variables.

Let $X : \Omega \to \mathbb{R}$ be a RV.

and $f : \mathbb{R} \to \mathbb{R}$ be a fn.

Now we are interested in

$$Y = f(X).$$

For example, $f(t) = t^2$.

$$Y = X^2$$

$$Y : \Omega \to \mathbb{R}$$

$$\omega \longmapsto Y(\omega) = f(X(\omega)) = X(\omega)^2$$

Not all $f: \mathbb{R} \to \mathbb{R}$ give $f(X)$ as R.V.

Q. which functions f give me $f(X)$ as RV?

A. Measurable functions.

Def. A fn $f: \mathbb{R} \to \mathbb{R}$ is said to be measurable if $f^{-1}(B) \in \mathcal{B}(\mathbb{R})$ if $B \in \mathcal{B}(\mathbb{R})$

Exp. Suppose X is $\bar{B}_{n,p}$ distributed, that is,

$$P(X=k) = \binom{k-1}{k-n} p^n (1-p)^{k-n}.$$

Let $Y = X - n$.
Which probability distribution does Y possess?

As.

$$P(Y=k)$$

$$= P(X=k+n)$$

$$= \binom{n+k-1}{k} p^n (1-p)^k$$

$$= \binom{-n}{k} p^n (p-1)^k.$$

$$k=0,1,\dots$$

Interpretation:

We perform a series of random trials where each time we may obtain either failure or success. Thereby, the success. prob. is p. Then the event $\{Y=k\}$ occurs iff if we observe the nth success in trial $k+n$.