Probability & Statistics

L7

Random variable.

RE, $(\Omega, \mathcal{A}, P)$

$$X : \Omega \longrightarrow \mathbb{R}$$

s.t. $B \in \mathcal{B}(\mathbb{R})$, $\underline{X^{-1}(B) \in \mathcal{A}}$.

$\mathcal{B}(\mathbb{R})$ = the σ-field generated by closed and bdd intervals $[a,b]$, $a<b$.

$$X^{-1}(B) = X^{-1}([a,b])$$

$$= \{\omega \in \Omega \mid a \le X(\omega) \le b\}$$

$\Omega \rightarrow$ set of all students in this class

$X : \Omega \rightarrow \mathbb{R}$

$\omega \longmapsto X(\omega) =$ the height of ω.

$a = 5$ ft, $b = 5.5$ ft.

$X^{-1}([5, 5.5]) =$ set of all students whose height lies betw 5 to 5.5

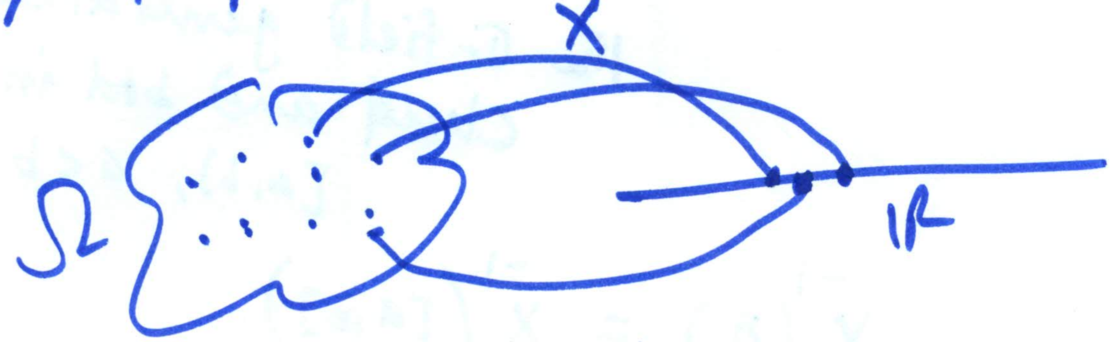
If $X^{-1}(B) \notin \mathcal{A}$ then $P(X^{-1}(B))$ is not defined.

For a discrete sample space:

$\Omega \to$ finite or countably infinite.

$\mathcal{A} = \mathcal{P}(\Omega)$

$\Rightarrow$ Any function $X : \Omega \to \mathbb{R}$



is a random variable

$$X^{-1}(B) \subseteq \Omega$$

$$\in \mathcal{A} = \mathcal{P}(\Omega)$$

$X \longrightarrow$ a discrete random variable.

| it can take only countably many values.

Given $x \in \mathbb{R}$,

$$X = x \equiv \{\omega \in \Omega \mid X(\omega) = x\}.$$

Define a function $p : \mathbb{R} \to \mathbb{R}$

s.t. $p(x) = P(X = x)$

$$\boxed{p(x) \geq 0, \quad \sum_{x \in \mathbb{R}} p(x) = 1.}$$

probability mass function corresponding to the discrete random variable X.

Exp. Binomial dist. $P(X=x) = p(x)$

Given a random variable X on a discrete sample space.

$$P_X(B) = P(\{\omega \in \Omega \mid X(\omega) \in B\}).$$

$B \subseteq \mathbb{R}$.

Exp. Toss a fair coin labelled on one side by '0' & other side by '1' three times.

$$\Omega = \{0,1\}^3$$

Let $X : \Omega \to \mathbb{R}$

$$X(\omega) = X((\omega_1, \omega_2, \omega_3))$$

$$= \omega_1 + \omega_2 + \omega_3$$

Then $P_X(\{0\}) = P(X=0)$

$$= P(\{(0,0,0)\}) = \frac{1}{8}$$

$$P_X(\{1\}) = P(X=1)$$

$$= P(\{(0,1,0), (1,0,0), (0,0,1)\})$$

$$= \frac{3}{8}$$

$$P_X(\{2\}) = P(X=2)$$

$$= P(\{(1,1,0), (0,1,1), (1,0,1)\})$$

$$= 3/8$$

$$P_X(\{3\}) = P(X=3) = P(\{(1,1,1)\})$$

$$= 1/8$$

Special Discrete Random variables.

① Poisson Distribution.

$\Omega = \{0, 1, 2, \ldots\} \equiv \mathbb{N}_0$

$\lambda \cancel{\geq} 0$, a given parameter.

$$\mathrm{Pois}_\lambda(\{k\}) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k \in \Omega$$

$$= P(X = k)$$

p.m.f. $\quad X_\lambda(\{x\}) \geq 0, \quad x \in \mathbb{R}$

$$\sum_{x \in \mathbb{R}} X_\lambda(\{x\}) = \sum_{k \in \Omega} X_\lambda(\{k\})$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= e^{-\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right)$$

$$= e^{-\lambda} \cdot e^{\lambda} = 1.$$

Poisson limit theorem.

Let $\{p_n\}$ be a seq. of numbers s.t. $0 \le p_n \le 1$ and $\lim_{n\to\infty} n p_n = \lambda$, for some $\lambda > 0$.

Then for all $k \in \mathbb{N}_0$,

$$\lim_{n\to\infty} B_{n,p_n}(\{k\}) = \mathrm{Pois}_\lambda(\{k\}).$$

Pf. $B_{n,p_n}(\{k\}) = \binom{n}{k} p_n^k (1-p_n)^{n-k}$

$$= \frac{n!}{k!\,(n-k)!} p_n^k (1-p_n)^{n-k}$$

$$= \frac{1}{k!}\left(\frac{n(n-1)\cdots(n-k+1)}{n^k}\right)(np_n)^k (1-p_n)^{n-k}$$

$$= \frac{1}{k!}\left(\frac{n}{n}\cdot\frac{n-1}{n}\cdot\ldots\cdot\frac{(n-k+1)}{n}\right)(np_n)^k (1-p_n)^{n-k}$$

$$(1-p_n)^{n-k} = \boxed{(1-p_n)^n}\cdot(1-p_n)^{-k}$$

$$\underbrace{\frac{n}{n}\to 1,\ \frac{n-1}{n}\to 1,\ \ldots,\ \frac{n-k+1}{n}\to 1}_{\text{as } n\to\infty} \qquad (np_n)^k \to \lambda^k \qquad (1-p_n)^{-k} \to 1$$

Recall: $\lim_{n\to\infty}\left(1+\frac{x_n}{n}\right)^n = e^x$

$$\lim_{n\to\infty}(1-p_n)^n = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

Conclusion:

(1) whenever n is large & p is small $B_{n,p} \approx Pois_\lambda$ where $\lambda \simeq np$.

(2) Poisson limit theorem explains why the Poisson distribution describes experiments with many trials and small success probability.

For example, if we look for a model for the # of car accidents per year, then Poisson distribution is a 'good' choice.

Hypergeometric distribution.

Among N delivered machines, M are defective. One chooses n of N machines randomly and checks them.

Q. What is the probability to observe 'm' defective machines in the sample

$$(N, M, n, m)$$

n machines out of N machines.

$$\binom{N}{n}$$

$\binom{M}{m}$ many ways to pick 'm' # of defective machines.

$\binom{N-M}{n-m}$ possibilities for non-defective.

$$\boxed{H_{N,M,n}(\{m\}) = \frac{\binom{M}{m}\binom{N-M}{n-m}}{\binom{N}{n}}, \quad 0 \le m \le n}$$

$$\parallel$$

$$P(X = m)$$

<u>H.W.</u> $\displaystyle\sum_{m=0}^{n} H_{N,M,n}(\{m\}) = n$

<u>Hint.</u> use Vandermonde's

Conclusion:

① Whenever n is large & p is small $B_{n,p} \approx Pois_\lambda$ where $\lambda = np$.

② Poisson limit theorem explains why the Poisson distribution describes experiments with many trials and small success probability.

For example, if we look for a model for the # of car accidents per year, then Poisson distribution is a 'good' choice.

Hypergeometric distribution.

Among N delivered machines, M are defective. One chooses n of N machines randomly and checks them.

Q. What is the probability to observe 'm' defective machines in the sample

$$\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}$$

Exp. In a pond are 200 fish. One day the owner of the pond catches 20 fish, marks them, and puts them back into the pond. After a while the owner catches once more 20 fish. Find the probability that among these fish there is exactly one marked.

Sol. $N = 200$, $M = 20$, $n = 20$

$$H_{200,20,20}(\{1\}) = ?$$

Remark. In an urn N balls. M – are white, $N-M$ are black. Choosing n balls out of the urn without replacing the choosen ones.

Then $H_{N,M,n}(\{m\})$ = the prob. to observe m white balls among n chosen

If we do the experiment without replacing the chosen balls, then this is a binomial distribution with success prob. $p = \frac{M}{N}$.

and hence the prob. for m white balls is

$$B_{n,\frac{M}{N}}(\{m\}) = \binom{n}{m}\left(\frac{M}{N}\right)^{m}\left(1-\frac{M}{N}\right)^{n-m}.$$

Obs. $\lim\limits_{\substack{N, M \to \infty \\ \frac{M}{N} \to p}} H_{N,M,n}(\{m\}) = B_{n,p}(\{m\})$

Consider $N = 10^{10}$
$M = 10^{6}$

Geometric Distribution.

At a first glance the model for the geometric distribution looks as that for the binomial.

In each trial we may observe '0' or '1', that is, failure or success.

while Binomial – the # of trials is fixed

Geometric – the # of trials is random.

We execute the experiment untill we observe success for the first time.

$$G_p(\{k\}) = p(1-p)^{k-1}, \; k \in \mathbb{N}$$
$$= P(X = k).$$

H.W. $\boxed{\sum_{k \in \mathbb{N}} p(1-p)^{k-1} = 1}$

Exp. Roll a die untill the first '6' shows up. What is the probability that this happens at an even number of trials.

Soln.

$$\sum_{K=1}^{\infty} G_{1/6}(\{2K\})$$

$$= \frac{1}{6} \sum_{K=1}^{\infty} \left(\frac{5}{6}\right)^{2K-1}$$

$$= 5/11.$$

Hoel, Port & Stone

Negative Binomial Distribution.

The geometric distribution describes the probability for having the first success in trial K.

Given a fixed $n \geq 1$, we ask for the probability that in trial K success appears not for the first time but for the n-th time.

$n > K$

Suppose $k \geq n$ & we have success in k-th trial.

Q. when is this the n-th one?

$\binom{k-1}{k-n}$ possibilities to distribute $k-n$ failures among first $k-1$ trials.

The prob. for n times success is p^n and for $k-n$ failures it is $(1-p)^{k-n}$.

$$B^{-}_{n,p}(\{k\}) = \binom{k-1}{k-n} p^n (1-p)^{k-n},$$

$$k = n, n+1, \ldots .$$

Exp. Roll a die successively. Determine the probability that in the 20th trial number '6' appears for the fourth time.

Sol: $p = \frac{1}{6}$, $n = 4$, $k = 20$.

Expected value of discrete RV.

Exp. Suppose N # of students ~~regi~~ registered this course and sit for certain exam. The # possible marks is 100. Given $j = 0, 1, \ldots, 100$, let n_j: # of studens who achieve marks j.

Now choose randomly (uniformly) one student. Name him/her ω. define $X(\omega) =$ the marks scored by ω.

Then X is a RV with values in $D = \{0, 1, \ldots, 100\}$.

Q. How X is distributed.

An expected value of $X = A$.

$$A = \frac{1}{N} \sum_{j \in D} j \cdot n_j = \sum j \cdot \frac{n_j}{N}$$

$$= \sum_{j \in D} j \, P(X = j)$$

Def. If X is a discrete RV with values in $\{x_1, x_2, \ldots\}$. Then the expected value of X is

$$E(X) = \sum_{j=1}^{\infty} x_j \, P(X = x_j)$$

Exp. (I) **Uniform distribut.**

$$E(X) = \frac{1}{N} \sum x_j$$

(II) **$B_{n,p}$.**

$$E(X) = \sum_{k=0}^{n} k \, P(X=k)$$

$$= \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)\,(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n} \frac{(n-1)!}{k!\,(n-1-k)!} p^{k} (1-p)^{n-k-1}$$

$$= np\,(p+(1-p))^{n-1}$$

③ Poisson RV.

$$E(X) = \sum_{K=0}^{\infty} K\, P(X=K)$$

$$= \sum_{K=1}^{\infty} K \cdot \frac{\lambda^K}{K!} e^{-\lambda}$$

$$= e^{-\lambda} \cdot \lambda \left(\sum_{K=1}^{\infty} \frac{\lambda^{K-1}}{(K-1)!} \right)$$

$$= \lambda.$$

④ Negative Binomial.

$$E(x) = \frac{n}{p}.$$

⑤ Geometric

$$E(x) = \frac{1}{p}.$$