Probability & Statistics

L6

Conditional Probability measure:

to analyze how occurance of an event influences the occurances of other events!

Recall, A = "sum of two outcomes is 5"
B = the first outcome is even

$$\boxed{P(A|B) = P(A)}$$

Independent Events:

Given two events A, B, we call them independent if

$$P(A|B) = P(A) \Leftrightarrow P(A\cap B) = P(A)P(B)$$

Note: Let A & B are independent.

Q. A & B^c are independent!!
or A^c & B^c are independent!!

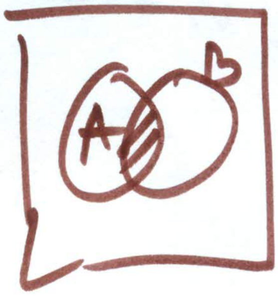
Let us prove that

A & B^c are independent!

$$P(A \mid B^c) = P(A)$$

$$\Leftrightarrow \quad \overset{\text{LHS}}{P(A \cap B^c)} = \overset{\text{RHS}}{P(A)\,P(B^c)}.$$

$$\underline{P(A)\,P(B^c)} = P(A)\,(1 - P(B))$$



$$= P(A) - P(A)\,P(B)$$

$$= P(A) - P(A \cap B)$$

$$= P(A \setminus (A \cap B))$$

$$= \underline{P(A \cap B^c)}$$

Let $A_1, A_2, \ldots, A_n$ be events.

<u>Pairwise indendent</u>:

A_i & A_j for any $i \neq j$, $i,j \in \{1, \ldots, n\}$ are independent.

<u>Independance of events</u>:

For any $I \subseteq \{1, 2, \ldots, n\}$

$$P\Big(\bigcap_{j \in I} A_j\Big) = \prod_{j \in I} P(A_j)$$

Exp. In an urn, there are n, $n \geq 2$ white balls and also n black balls. One chooses two balls without replacing the first one.

Let A be the event that the second ball is black while B occurs if the first ball was white.

Are A & B independent.

Sol^n.

$$P(B) = \frac{n}{2n} = \frac{1}{2}$$

$$P(A) = P(B)\,P(A|B) + P(B^c)\,P(A|B^c)$$

$$= \frac{1}{2} \cdot \frac{n}{2n-1} + \frac{1}{2} \cdot \frac{n-1}{2n-1}$$

$$= \frac{1}{2}.$$

$$\therefore \quad P(A)\,P(B) = 1/4$$

$$P(A \cap B) = P(B)\,P(A|B) = \frac{1}{2} \cdot \frac{n}{2n-1}$$

$$= \frac{n}{4n-2} \neq \frac{1}{4}$$

When 'n' is large A & B tend to be independent.

Let $A_1, A_2, \ldots, A_n$ be independent.
Define a collection $B_1, B_2, \ldots, B_n$
where $B_i = A_i$ or A_i^c $\forall i$

Then $B_1, \ldots, B_n$ are also independent.

Pf. Simple using the fact for
A & B and A & B^c.

Ex. A machine consists of n components. These components break down with probabilities $p_1, p_2, \ldots, p_n$. Moreover, assume that they ~~ar~~ break down independently of each other. Find the probability that chosen machine stops working.

Sol^n. Assumption: Break down of atleast one component implise that the machine stops working!

$A_j \rightarrow$ jth component breaks down

A_j, $j=1:n$ are indepndt.

$M \longrightarrow$ machine stops working.

$$P(M) \qquad M = \bigcup_{j=1}^{n} A_j$$

$$= 1 - P(M^c)$$

$$= 1 - P\left(\bigcap_{j=1}^{n} A_j^c\right)$$

$$= 1 - \prod_{j=1}^{n} P(A_j^c)$$

$$= 1 - \prod_{j=1}^{n} (1-p_j).$$

Assumption: Breaking down of all the components implies that the machine stops working.

$$M = \bigcap_{j=1}^{n} A_j$$

$$P(M) = P\left(\bigcap_{j=1}^{n} A_j\right)$$

$$= \prod_{j=1}^{n} p_j \ .$$

Random Variable

R.E. $(\Omega, \mathcal{A}, P)$

Let us try to define a map

$$X : \Omega \longrightarrow \mathbb{R}$$

motivation: associate numbers with outcomes.

If $\omega \in \Omega$ occurs, $X(\omega)$

occurrance of ω is uncertain

Ex. Toss a fair coin 'n' times.
label the sides of the coin as '1' & '0'
Then $\Omega = \{0,1\}^n$, $P \to$ uniform pm.

Let $X : \Omega \longrightarrow \mathbb{R}$

$$X(\omega) = X((\omega_1, \omega_2, \ldots, \omega_n))$$
$$= \omega_1 + \omega_2 + \cdots + \omega_n$$

Exp. Roll a fair die twice.

$$\Omega = \{(\omega_1, \omega_2) \mid \omega_1, \omega_2 \in \{1, \dots, 6\}\}$$

Define $X : \Omega \to \mathbb{R}$ as

1. $X(\omega) = \max\{\omega_1, \omega_2\}$
2. $X(\omega) = \min\{\omega_1, \omega_2\}$
3. $X(\omega) = \omega_1 + \omega_2$

$(\Omega, \mathcal{A}, P)$

Let $A \in \mathcal{A}$

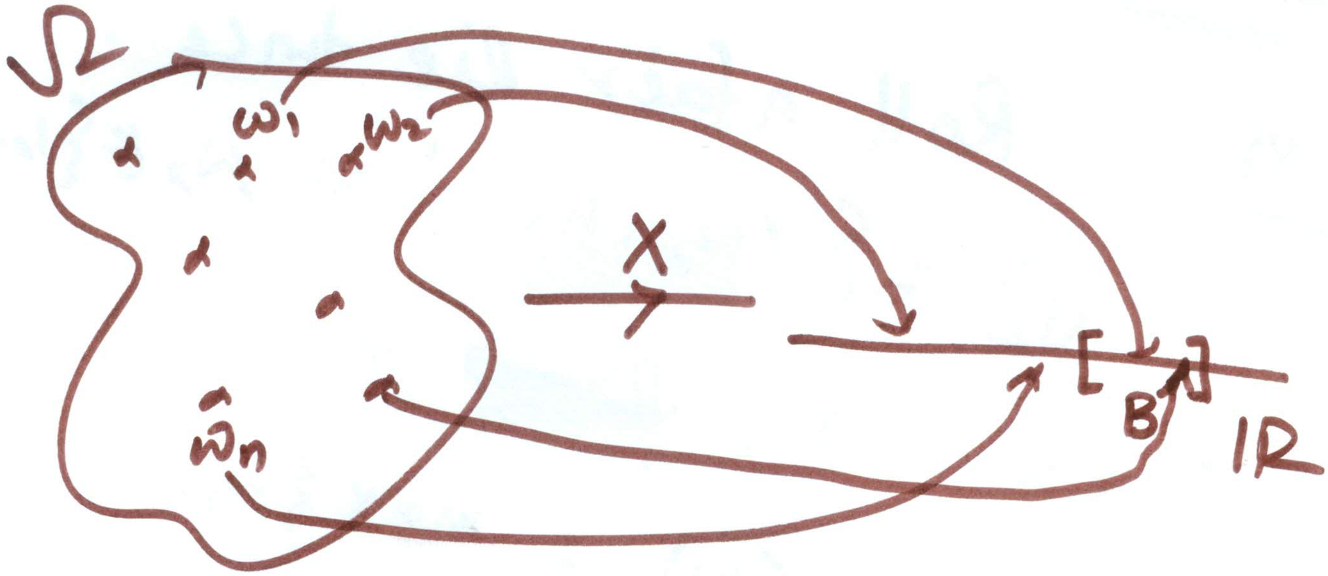
If $\omega \in A$ occurs then A occurs.

$X : \Omega \to \mathbb{R}$.

Let $B \subseteq \mathbb{R}$, $B \in \mathcal{B}(\mathbb{R})$ ← Borel σ-field

Then we observe $\omega \in \Omega$ for which $X(\omega) \in B$ or equivalently

$$\{X \in B\} = \{\omega \in \Omega \mid X(\omega) \in B\}.$$



$$X^{-1}(B) = \underbrace{\{\omega \in \Omega \mid X(\omega) \in B\}}_{\text{event}} \subseteq \Omega$$

Thus $\omega \in \Omega$ occurs if and only if $X(\omega) \in B$.

<u>Defn.</u> Let $(\Omega, \mathcal{A}, P)$ ps.
A mapping $X : \Omega \to \mathbb{R}$ is called a random variable if $B \in \mathcal{B}(\mathbb{R}) \Rightarrow X^{-1}(B) \in \mathcal{A}$.