Probability & Statistics

L5

Discrete Probability space

$p(a_k) = 1/n$ — Uniform distribution, $\Omega = \{a_1, \ldots, a_n\}$

$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$, — Binomial distribution, $\Omega = \{0, 1, \ldots, n\}$

p – prob. of success

$1-p$ – prob of failure.

Conditional Probability

Exp. Roll a dice twice. The prob of the event "sum of both rolls equals 5". Suppose we were told that the first roll was an even number.

Does this additional information makes the event "sum equal to 5" more likely? or does it even diminish the probability of the event?

— restricting the sample space.

B → sample space ≡ the first roll is an even number.

A = Sum of the outcomes equal to 5

Then $P(A \cap B) = ?$

$A, B \in \mathcal{P}(\Omega)$ where Ω is the original sample space!

In both the cases $P(A) = \frac{1}{9}$, so the additional information does not change the probability.

A = Sum of outcomes equals to 6

$P(A) = 5/36$

$P(A \cap B) = 1/9$

Def^n: Let $(\Omega, \mathcal{A}, P)$ be a probability space. Given $A, B \in \mathcal{A}$ with $P(B) > 0$, the prob. of A under the condition B is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \cdot \left(\frac{\#(A \cap B)}{\#(B)} \right.$$

when P is uniform)

Exp. In an urn there are two white and two black balls. Choose two balls without replacing the first one. Evaluate the probability of occurance of a black ball in first draw and of a white in the second one:

Ans. $1/3$.

Exp. Among three non-distinguishable coins are two fair coins and one is biased. Tossing the biased coin 'head' appears with prob. $\frac{1}{3}$, and 'tail' appears . . . . $2/3$.
We choose a random one out of these three coins and toss it.
Find the the prob. to observe 'tail' at the biased coin.

Ans. $2/9 = \frac{1}{3} \cdot \frac{2}{3}$

$\Omega \to$ sample space

$B \in \mathcal{P}(\Omega)$, $P(B) > 0$

$P(A|B)$

$=$

$$P : \mathcal{A} \to [0,1]$$

$$A \mapsto P(A|B).$$

Check that P is a valid probability measure:

(i) $P(\emptyset) = 0$. $P(\Omega) = 1$

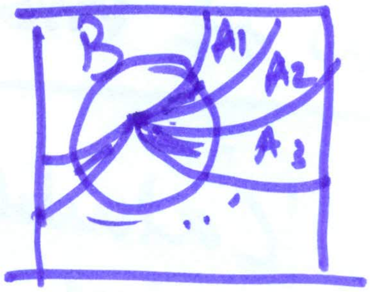
$$P(\emptyset|B) = \frac{P(\emptyset \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = 0$$

$$P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

(ii) $$P\left(\bigcup_{j=1}^{\infty} A_j \,\middle|\, B\right) = \sum_{j=1}^{\infty} P(A_j|B)$$

when A_j's are disjoint.

$$P\left(\bigcup_{j=1}^{\infty} A_j \mid B\right)$$



$$= \frac{P\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \cap B\right)}{P(B)}$$

$$= \frac{P\left(\bigcup_{j=1}^{\infty} (A_j \cap B)\right)}{P(B)},$$

since A_js are disjoint, $A_j \cap B$ are disjoint.

$$= \frac{\sum_{j=1}^{\infty} P(A_j \cap B)}{P(B)}$$

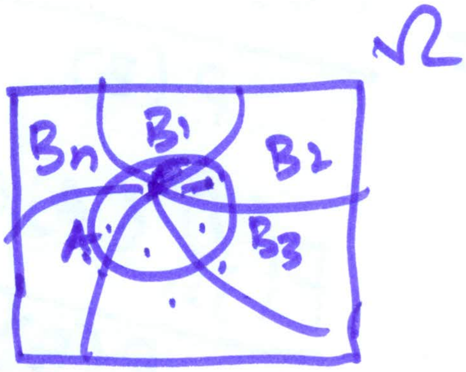
$$= \sum_{j=1}^{\infty} \frac{P(A_j \cap B)}{P(B)}$$

$$= \sum_{j=1}^{\infty} P(A_j \mid B).$$

This proves that conditional probability is a valid probability measure.

Law of Probability.

'Calculate prob. of an event using conditional probabilities of events'



Let $(\Omega, \mathcal{A}, P)$ be a PS and $B_1, B_2, \dots, B_n \in \mathcal{A}$ s.t. they are disjoint and $\bigcup_{j=1}^{n} B_j = \Omega$.

Then for each $A \in \mathcal{A}$

$$P(A) = \sum_{j=1}^{n} P(B_j)\, P(A \mid B_j)$$

Pf.

$$\begin{aligned} P(A) &= P(A \cap \Omega) \\ &= P\Big(\bigcup_{j=1}^{n} B_j \cap A\Big) \\ &= P\Big(\bigcup_{j=1}^{n} (B_j \cap A)\Big) \\ &= \sum_{j=1}^{\infty} P(B_j \cap A) \\ &= \sum_{j=1}^{\infty} P(B_j)\, P(A \mid B_j) \end{aligned}$$

Exp. A fair coin is tossed four times. Suppose we observe exactly k times 'heads' for some $k = 0, 1, 2, 3, 4$.

According to the observed k, we take k dice and roll them.

Find the probability that number '6' does not appear.

Note that $k=0$ means that we do not roll a dice, hence in this case '6' cannot appear.

Ans. $\left(\frac{11}{12}\right)^4 = 0.706066743$

$$\Omega = \{ (k,Y), (k,N) \mid 0 \le k \le 4 \}.$$

Let $N = \{ (0,N), \ldots, (4,N) \}$

and $B_k = \{ (k,Y), (k,N) \}$, $k = 0, \ldots, 4$.

$P(N|B_0) = 1$, $P(N|B_1) = 5/6$, $P(N|B_2) = \left(\frac{5}{6}\right)^2$

$\ldots$ $P(N|B_4) = (5/6)^4$.

$$P(B_k) = \binom{4}{k} \frac{1}{2^4}, \quad k = 0, \ldots 4.$$

$$P(N) = \sum_{k=0}^{4} P(N|B_k)\, P(B_k)$$

Exp. Three different machines M_1, M_2 and M_3 produce light blubs. In a single day, M_1 produces 500 blubs, $M_2 \to 200$, $M_3 \to 100$. The quality of the produced blubs depends on the machines. Among the light blubs produced by M_1 are 5% defective, M_2 10%, M_3 only 2%.

At the end of the day, a controller chooses by random one of the 800 produced light blubs and test it. Determine the probability that the blub is defective.

Ans. $0.05875 = \frac{47}{800}$

Defn. Let $(\Omega, \mathcal{A}, P)$ and there are disjoint events $B_1, .., B_n \in \mathcal{A}$ s.t. $\Omega = \bigcup_{j=1} B_j$. Then we call $P(B_1), \dots, P(B_n)$ "a priori probabilities" of $B_1, \dots, B_n$.

Let $A \in \mathcal{A}$ with $P(A) > 0$. Then the conditional probabilities $P(B_j | A), j = 1:n$ are said to be "a posteriori probabilities".

Q. How are these probabilities related?

Bayes' formula.

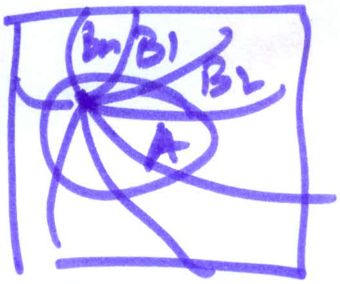
Suppose we are given disjoint events $B_1, \ldots, B_n$ satisfying $\bigcup_{j=1}^{n} B_j = \Omega$ and $P(B_j) > 0$.

Let A be an event with $P(A) > 0$.

Then for each $j \leq n$,

$$P(B_j | A) = \frac{P(B_j)\, P(A|B_j)}{\sum_{i=1}^{n} P(B_i)\, P(A|B_i)}$$

Pf. $$P(B_j | A) = \frac{P(B_j \cap A)}{P(A)}$$



$$P(A) = P(A \cap \Omega)$$

$$P(B_j \cap A) = P(A|B_j)\, P(B_j)$$

Exp. In order to figure out whether or not a person suffers from a certain disease, say disease X, a test is assumed to give a clue.

If the tested person is sick, then the test is +ve in 96% of cases.

If the person is well, then with 94% accuracy the test is negative.

Furthermore, it is known that 0.4% of the population suffer from X.

Now, a person chosen by random, is tested. Suppose the result is +ve. find the prob. that this person really suffers from X.

Ans. 0.0603774.

$\Omega = \{(X, p), (X, n), (X^c, p), (X^c, n)\}$.

& $A = \{(X, p), (X^c, p)\}$.

& $B = \{(X, p), (X, n)\}$

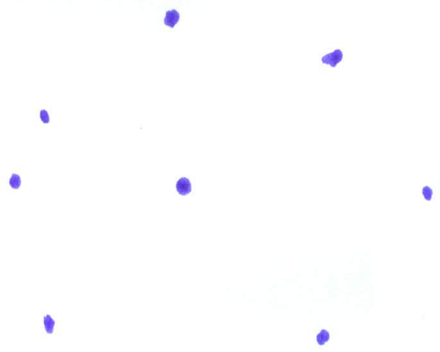
~~$P(B|A)$~~ $P(A|B) = .96$

$P(A|B^c) = .06$, $P(B) = .004$

$P(B^c) = 0.996$

$P(B|A) =$ ← Bayes' Formula.

H.W.



Given N # of students.

$p \rightarrow$ the probability that two students get acquainted.

Assumption: 'p' is fixed for any pair of distinct students.

Q. What is the probability that a student will have 'k' acquaintances after end of this course?

Ans. $\Omega = \{(i,j) \mid i \neq j\}$

$$\binom{N-1}{k} p^{k} (1-p)^{N-1-k}$$

Q What is the probability that the network will be connected?