Probability & Statistics

L4

Discrete Probability space.

RE — Ω — countable

$\mathcal{A}$ — $P(\Omega)$, $\#(\mathcal{A}) = 2^{\#(\Omega)}$

P — PM

$$\Omega = \{ \omega_1, \omega_2, \ldots \}$$

$$p_i = P(\omega_i)$$

Any such PM,

$$\boxed{p_i \geq 0, \quad \sum_i p_i = 1}$$

We are interested to study special PMs which can model many real world random experiments.

Uniform PM

$$\Omega = \{\omega_1, \dots, \omega_N\}$$

$$p_i = P(\{\omega_i\}) = \frac{1}{N}$$

<u>Exp2</u> A <u>fair coin</u> is labelled with '0' and 1. Toss it n-times.

<u>Q.1</u> what is the probability that for some fixed $i \le n$, the ith toss equals '0'?

<u>Ans.</u> $\Omega = \{(\omega_1, \dots, \omega_n) \mid \omega_i \in \{0,1\}\}$

$$\#(\Omega) = 2^n$$

$$A = \{(\omega_1, \dots, \omega_{i-1}, 0, \omega_{i+1}, \dots, \omega_n) \mid \omega_j \in \{0,1\}\}$$

$$\#(A) = 2^{n-1}$$

$$P(A) = \frac{2^{n-1}}{2^n} = \frac{1}{2}.$$

Q2. What is the probability that we observe $k \le n$ times the number '1'?

Ans. $\dfrac{\binom{n}{k}}{2^n}$? ✓

$\dfrac{\binom{n}{k}}{2^{2n}}$ XX

Binomial Distribution.

$\Omega = \{0, 1, \dots, n\}$.

$0 \le p \le 1$

$$p_k = P(\{k\}) \;, \; k \in \Omega$$
$$= \binom{n}{k} p^k (1-p)^{n-k}$$

Check. $p_k \geq 0$, $\displaystyle\sum_{k=0}^{n} p_k = 1$

$$p_k = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, .., n$$

$p_k \geq 0$?

$\sum p_k = 1$

i.e. $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$

$= (p + 1 - p)^n$

$= 1$

$(a+b)^n$

$= !!$

Q. What RE does this PM can explain !!

Ans. Assume $n = 1$.

$\Omega = \{0, 1\}$

$P(\{0\}) = \binom{1}{0} p^0 (1-p)^1 = 1 - p$

$P(\{1\}) = \binom{1}{1} p^1 (1-p)^0 = p$

Any RE with two possible outcomes exp. success & failure can be modelled by this PM, where success ('1') occurs with probability 'p' and failure ('0') occurs with probability $1-p$.

$n = 2$, $\Omega = \{0, 1, 2\}$

$p(\{0\}) = P\{\text{'0' times success out of 2 trials}\}$

$p(\{1\}) = P\{\text{1 time success out of 2 trials}\}$

$p(\{2\}) = P\{\text{2 times of success out of 2 trials}\}$

For any $k \leq n$, this p_k can explain the success of one possibility 'k' times !!!

$$p_k = \binom{n}{k} p^k (1-p)^{n-k}$$

'observing exactly k-times success or n-k times failure'

For any event $A \subseteq \Omega$

$$\boxed{P(A) = \sum_{k \in A} \binom{n}{k} p^k (1-p)^{n-k}}$$

This PM ~~is~~ denoted as $\mathcal{B}_{n,p}$, called binomial distribution (PM).

xp. An exam consists of 100 questions where each of the question may be answered either with 'yes' or 'no'. To pass the exam at least 60 questions have to be answered correctly. Let 'p' be the probability to answer a single question correctly. How big has p to be in order to pass the exam with a probability greater than 75%?

Ans.

$$\sum_{K=60}^{100} \binom{100}{K} p^{K} (1-p)^{100-K} \geq 0.75$$

$$p \geq 0.62739$$

$\Rightarrow$ ~~[illegible]~~.

~~Exp2. A fair coin~~

Exp2. In a class there are N students. Find the probability that at least two of them have their birthday on Jan 16!!.

Ans. $p = \frac{1}{365}$ | no leap year, no twins

$$\sum_{K=2}^{N} \binom{N}{K} \left(\frac{1}{365}\right)^{K} \left(\frac{364}{365}\right)^{N-K},$$

$$= 1 - B_{N,\frac{1}{365}}(\{0\}) - B_{N,\frac{1}{365}}(\{1\})$$

$$= 1 - \left(\frac{364}{365}\right)^{N} - \frac{N}{365}\left(\frac{264}{365}\right)^{N-1}$$

For $N = 199$, $\underline{p = ??}$