(Sec-1) **Probability & Statistics** **L3**

Random Experiments. (RE)

$(\Omega, \mathcal{A}, P)$

$\downarrow$

(σ-field)

$\downarrow$

some events are more likely than others.

Need & the concept of σ-field

Exp. 1 height.

$[5,6] \subseteq [4, 6.5]$

$P: \mathcal{A} \rightarrow [0,1]$

Exp.

$\rightarrow$ throwing a dart.

RE $\to$ Ω is countable | Discrete

$\to$ Ω is uncountable | Continuous

Discrete Probability measure.

Ω — Contable.

$$\mathcal{A} = \mathcal{P}(\Omega)$$

$$P : \mathcal{P}(\Omega) \longrightarrow [0,1]$$

(i) $P(\Omega) = 1$,
$P(\phi) = 0$

(ii) P is σ-additive

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j)$$

when A_j's ~~are~~ is a disjoint collection

Q. What are all possible probability measures on a discrete sample space!!

Let $\Omega = \{\omega_1, \dots, \omega_N\}$

$\mathcal{A} = \mathcal{P}(\Omega)$

Let $P : \mathcal{A} \to [0,1]$ any probability measure defined on $\mathcal{A}$.

Obs. If P is a probability measure on $\mathcal{P}(\Omega)$ then the numbers p_j,

$$p_j = P(\{j\}) = P(\{\omega_j\})$$

satisfy the following

$$0 \le p_j \le 1, \quad \sum_{j=1}^{N} p_j = 1$$

$$\Omega = \bigcup_{j=1}^{N} \{\omega_j\}$$

$$\Rightarrow \quad P(\Omega) = P\left(\bigcup_{j=1}^{N} \{\omega_j\}\right)$$

$$= \sum_{j=1}^{N} P(\{\omega_j\})$$

$$1 = p_1 + p_2 + \cdots + p_N$$

$$\Rightarrow \quad \boxed{1 = \sum_{j=1}^{N} p_j}$$

Then $A \in \mathcal{A}$

$A \subseteq \{\omega_1, \ldots, \omega_N\}$ $\quad \| \quad = \bigcup_{j=i}^{K} \{\omega_j\}$

$$\Rightarrow \quad P(A) = P(\{\omega_j\}) = p_i + p_{i+1} + \cdots + p_K$$

If P is a probability measure

$\Downarrow$

$$\boxed{0 \le p_j \le 1 \text{ and } \sum_{j=1}^{N} p_j = 1}$$

$$\Omega = \{\omega_1, \ldots, \omega_N\}$$
$$\downarrow \; \downarrow \; \downarrow \; \ldots \; \downarrow$$
$$p_1 \; p_2 \; p_3 \ldots p_N$$

$$\left\{\begin{array}{l}\text{Set of all probability} \\ \text{measures on} \\ \Omega = \{\omega_1, \ldots, \omega_N\}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} p_1, \ldots, p_N, \\ 0 \le p_j \le 1 \\ \sum p_j = 1 \\ j = 1, \ldots, N \end{array}\right\}$$

Instead of finite Ω, if I have $\Omega = \{1, 2, \ldots\}$, a countably infinite set

If P is a probability measure

$\Downarrow$

$$\boxed{0 \le p_j \le 1 \quad \text{and} \quad \sum_{j=1}^{N} p_j = 1}$$

$$\Omega = \{\omega_1, \ldots, \omega_N\}$$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \cdots \downarrow \\ p_1 & p_2 & p_3 & \ldots p_N \end{array}$$

$\left\{ \text{Set of all probability measures on } \Omega = \{\omega_1, \ldots, \omega_N\} \right\} \longleftrightarrow \left\{ p_1, \ldots, p_N, \ 0 \le p_j \le 1, \ \sum p_j = 1, \ j = 1, \ldots, N \right\}$

Instead of finite Ω, if I have $\Omega = \{1, 2, \ldots\}$ a countably infinite set

Then, $\Omega = \{\omega_1, \omega_2, \ldots\}$

$$\mathcal{A} = \mathcal{P}(\Omega)$$

$$P : \mathcal{A} \to [0,1]$$

$$\Updownarrow$$

$\{p_1, p_2, \ldots\}$, such that

$$0 \le p_j \le 1, \quad \sum_{j=1}^{\infty} p_j = 1$$

So for any $A \subseteq \Omega$

$$P(A) = \sum_{\omega_j \in A} P(\{\omega_j\})$$

<u>Exp. 1</u> ~~$\Omega = \mathbb{N} = \{1, 2, \dots\}$~~

Rolling a Die.

$$\Omega = \{1, 2, \dots, 6\}.$$

Any PM (probability measure) is nothing but, $p_1, p_2, \dots, p_6$

s.t. $0 \le p_j \le 1$, $p_1 + p_2 + \dots + p_6 = 1$.

— Think Geometrically.

<u>Exp. 2</u>

$$\Omega = \mathbb{N} = \{1, 2, \dots\}$$

$$\begin{matrix} 1 & 2 & \dots \\ \downarrow & \downarrow & \dots \\ p_1 & p_2 & \end{matrix}$$

$0 \le p_j \le 1$, $\sum_{j=1}^{\infty} p_j = 1$.

Define $p_j = P(\{j\}) = 2^{-j}$, $j = 1, 2, \dots$

$$\sum_{j=1}^{\infty} \frac{1}{2^j} = 1$$

$$P(\underbrace{\{2,4,6,\dots\}}_{}) \quad \{2,4,6,\dots\} = 2\mathbb{N}$$

$$= \sum_{j \in 2\mathbb{N}} P(\{j\})$$

$$= \sum_{j=1}^{\infty} \frac{1}{2^{2j}}$$

$$= \frac{1}{3}$$

Exp. 3. $\Omega = \mathbb{Z} \setminus \{0\}$

Define a PM.

For each $j \in \{-1, 1, \cancel{0}, 2, -2, \ldots$

$$p_j = P(\{j\}).$$

Let $p_j = \frac{c}{j^2}$, $c > 0$

$j \in \mathbb{Z} \setminus \{0\}$

since $\sum p_j = 1$

$$\Rightarrow \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{c}{j^2} = 1$$

$$\Rightarrow c \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{1}{j^2} = 1$$

$$\Rightarrow 2c \sum_{j=1}^{\infty} \frac{1}{j^2} = 1$$

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$$

$$\Rightarrow \quad 2c \sum_{j=1}^{\infty} \frac{1}{j^2} = \cancel{\pi^2}\ 1$$

$$\Rightarrow \quad \boxed{c = \frac{3}{\pi^2}}$$

$$p_j = \frac{3}{\pi^2 j^2}$$

Q. $P(\{1,2,3,\ldots\})$

$$= \sum_{j=1}^{\infty} \frac{3}{\pi^2 j^2} = \frac{1}{2}$$

Q. $P(\{2,4,6,\ldots\}) = \frac{1}{8}.$

Specific Discrete Probability Measures.

(I) Uniform Distribution on a finite set.

$$\Omega = \{\omega_1, \ldots, \omega_N\}$$

$$\mathcal{A} = \mathcal{P}(\Omega),$$

$$P(\{\omega_j\}) = \frac{1}{N}, \ \forall j$$

For any event $A \subseteq \Omega$,

$$P(A) = \frac{\#(A)}{N},$$

← number of elements in A.

Exp 1. In a lottery, 6 numbers are chosen out of 49 and each number appears only once. What is the probability that the chosen numbers are exactly the six ones on my lottery coupon.

Ans.

$$\boxed{\frac{6!}{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}}$$

$$= \frac{1}{\binom{49}{6}}$$

$$= 7.15112 \times 10^{-8}$$