Probability & Statistics

L-18

Prob. Let X & Y have the joint pdf

$$p(x,y) = \exp\left(c + 4x + 4y - \frac{x^2}{2} - \frac{y^2}{2} - \frac{x^2y^2}{2}\right)$$

where c is a constant.

Q. Find $P_{X|Y}(x|y)$.

write this in the form. $e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Ans. $$p_Y(y) = \int_{-\infty}^{\infty} p(x,y)\,dx$$

$$= \exp\left(c+4y-\frac{y^2}{2}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{(1+y^2)}{2}x^2+4x\right)dx$$

$$\vdots$$

$$= \frac{\sqrt{2\pi}}{\sqrt{1+y^2}}\exp\left(c+4y-\frac{y^2}{2}+\frac{8}{1+y^2}\right)$$

$$p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$$

$$\vdots$$

$$\sim N\left(\frac{4}{1+y^2}, \frac{1}{1+y^2}\right)$$

Observation This is an example where conditional densities are normal but the joint pdf ~~are~~ is not bivariate normal

F-distribution & t-distribution.

Recall Gamma distribution.

$$p_X(x;\alpha,\beta) = \frac{\beta(\beta x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x}$$

$$= \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, \quad x \geq 0$$ ✓

my earlier defn.

$$p_X(x) = \frac{1}{\alpha^\beta \Gamma(\beta)} x^{\beta-1} e^{-x/\alpha}$$

$$E(X^k) = \frac{(\alpha+k-1)\ldots\alpha}{\beta^k}$$

$$E(X) = \frac{\alpha}{\beta}, \quad E(X^2) = \frac{(\alpha+1)\alpha}{\beta^2}$$

$$\operatorname{Var}(X) = \frac{\alpha}{\beta^2}.$$

Prop. Let $X_1, X_2, \ldots, X_n$ be a seq. of independent random variables

$X_1 \sim \Gamma(\alpha_1,\beta)$, $X_2 \sim \Gamma(\alpha_2,\beta), \ldots, X_n \sim \Gamma(\alpha_n,\beta)$

then $X_1 + X_2 + \cdots + X_n \sim \Gamma(\alpha_1+\alpha_2+\cdots+\alpha_n, \beta)$

MGF of $\Gamma(\alpha, \beta)$

$$E(e^{tX}) = \int_0^\infty e^{tx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx$$

$$= \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\beta-t)x} dx$$

$$= \frac{\beta^\alpha}{(\beta-t)^\alpha} \int_0^\infty \underbrace{\frac{(\beta-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\beta-t)x}}_{\Gamma(\alpha,\, \beta-t)} dx$$

$$= \left(\frac{\beta}{\beta-t}\right)^\alpha .$$

Calculate the mgf of $X_1 + X_2 + \cdots + X_n$.

$$E\left(e^{t\sum_{j=1}^n X_j}\right) = E\left(\prod_{j=1}^n e^{tX_j}\right)$$

$$= \prod_{j=1}^n \left(\frac{\beta}{\beta-t}\right)^{\alpha_j}$$

$$= \left(\frac{\beta}{\beta-t}\right)^{\sum \alpha_j}$$

$$\therefore \sum_{j=1}^n X_j \sim \Gamma\left(\sum_{j=1}^n \alpha_j, \beta\right).$$

Recall: χ^2_n distribution

$$\chi^2_n = \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$$

If X_i is i.i.d. standard normal then

$$X_1^2 + X_2^2 + \cdots + X_n^2 \sim \chi^2_n.$$

Pf. Let $X \sim N(0,1)$. Then

$$P(X^2 \le x) \leftarrow \text{cdf of } X^2$$

$$= P(-\sqrt{x} < X < \sqrt{x})$$

$$= \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt.$$

Then the pdf of X^2 is

$$p_{X^2}(x) = \frac{d}{dx} \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

Leibniz rule

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{x})^2}{2}} (\sqrt{x})' - \frac{1}{\sqrt{2\pi}} e^{-\frac{(-\sqrt{x})^2}{2}} (-\sqrt{x})'$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{x}} e^{-x/2} = \frac{1}{\sqrt{2\pi}} x^{\frac{1}{2}-1} e^{-x/2}$$

$$\sim \Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$$

F - distribution

Let us consider two independent random variables

$$X \sim \chi^2_k = \Gamma\left(\frac{k}{2}, \frac{1}{2}\right)$$

$$Y \sim \chi^2_m = \Gamma\left(\frac{m}{2}, \frac{1}{2}\right)$$

Then the distribution of the random variable

$$Z = \frac{X/k}{Y/m}$$ is called

Fisher distribution with degrees of freedom k & m, denoted by $F_{k,m}$.

The pdf is given by

$$\frac{\Gamma\left(\frac{k+m}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{m}{2}\right)} t^{\frac{k}{2}-1} (1+t)^{-\frac{k+m}{2}}$$

student's
t - distribution.

Consider the random variable

$$T = \frac{X_1}{\sqrt{\frac{1}{n}\left(Y_1^2 + Y_2^2 + \cdots + Y_n^2\right)}}$$

where $X_1, Y_1, Y_2, \ldots, Y_n$ are i.i.d. ~~n.r.~~ standard normal.

The pdf of T is given by

$$\phi_T(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{1}{\sqrt{n}} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}$$