Probability and Statistics
Hints/Solutions to Assignment No. 2

1. (a) No, as it is not continuous from right at x = ½ .
   (b) No, $\lim_{x\to-\infty} F(x) \neq 0, \lim_{x\to\infty} F(x) \neq 1.$
   (c) Yes
   (d) Yes

2. $P\left(\frac{1}{2} \le X \le \frac{5}{2}\right) = F\left(\frac{5}{2}\right) - F\left(\frac{1}{2}-\right) = \frac{11}{24} - \frac{1}{8} = \frac{19}{24}.$

   $P(1 < X < 2) = F(2-) - F(1) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

   The random variable X is continuous in the intervals (0, 1) and (1, 2) with the uniform density f(x) = 1, and discrete at points 1, 2 and 3 with probabilities 1/4, 1/6 and 1/12 respectively. So

   $E(X) = \int_0^1 \frac{x}{4}\,dx + \int_1^2 \frac{x}{4}\,dx + 1.\frac{1}{4} + 2.\frac{1}{6} + 3.\frac{1}{12} = \frac{4}{3}.$

   $E(X^2) = \int_0^1 \frac{x^2}{4}dx + \int_1^2 \frac{x^2}{4}dx + 1^2.\frac{1}{4} + 2^2.\frac{1}{6} + 3^2.\frac{1}{12} = \frac{7}{3}.$ $V(X) = \frac{5}{9}.$ Median (X) = 1.

3. P(X = 1) = P(X = 2) = ¼, P(X = 3) = ½.

   $F_X(x) = 0, \quad x < 1,$
   $=1/4, \quad 1 \le x < 2,$
   $=1/2, \quad 2 \le x < 3,$
   $=1, \quad x \ge 3.$

4. Let S denote a survival and D denote a death of a guinea pig during the trial, then the sample space for this experiment can be described by
   Ω = {SS, SDS, SDD, DSS, DSD, DDSS, DDSD, DDDS, DDDD} and the probabilities associated with these sample points are given by 4/9, 4/27, 2/27, 4/27, 2/27, 4/81, 2/81, 2/81, 1/81 respectively.

   Let X = the number of survivors, Y = number of deaths. Then the pmf of X is
   P(X = 2) = P({SS, SDS, DSS, DDSS}) = 64/81,
   P(X = 1) = P({SDD, DSD, DDSD, DDDS}) = 16/81,
   P(X = 0) = P({DD}) = 1/81,
   and the pmf of Y is
   P(Y = 0) = P({SS}) = 4/9, P(Y = 1) = P({SDS, DSS}) = 8/27,
   P(Y = 2) = P({SDD, DSD, DDSS}) = 16/81,
   P(Y = 3) = P({DDSD, DDDS}) = 4/81, P(Y = 4) = P({DDDD}) = 1/81.

5. $X \sim Bin(3, \frac{1}{2}), \quad Y \sim P(\lambda).$

   So P(X + Y = 1)
   = P(X = 0, Y = 1) + P(X = 1, Y = 0)
   = P(X = 0) P(Y = 1) + P(X = 1) P(Y = 0), as X and Y are independent
   $= \left(\frac{1}{2}\right)^3 .e^{-1} + 3.\left(\frac{1}{2}\right)^3 .e^{-1} = \frac{e^{-1}}{2} = 0.1839.$

6. The cdf is given by

$$
\begin{aligned}
F_X(x) &= 0, && \text{if } x < 0,\\
&= x^2/4, && \text{if } 0 \le x < 1,\\
&= (2x-1)/4, && \text{if } 1 \le x < 2,\\
&= (6x - x^2 - 5)/4, && \text{if } 2 \le x < 3,\\
&= 1, && \text{if } x \ge 3.
\end{aligned}
$$

E(X) = 3/2, Median (X) = 3/2, Var(X) = 5/12.

7. Let X = the number of second generation particles,
   Let Y = the number of third generation particles,
   Then X → 1, 2, 3; Y →1, 2, …, 9.
   P(Y = 1) = 1/9, P(Y = 2) = 4/27, P(Y = 3) = 16/81, P(Y = 4) = 4/27 = P(Y = 5),
   P(Y = 6) = 10/81, P(Y = 7) = 2/27, P(Y = 8) = 1/27, P(Y = 9) = 1/81.
8. Let X be the number of tests required. Then X is either 1 or 11.
   P(X =1) = P(none has the disease) = $(0.99)^{10}$,
   P(X = 11) = P(at least one has disease) = 1 – $(0.99)^{10}$,
   E(X) = 11- $10(0.99)^{10}$.
9. $P(X = i) = \dfrac{n_i}{m}, \quad i = 1,....,r, \qquad E(X) = \sum_{i=1}^{r} \dfrac{in_i}{m}.$

   $$P(Y = i) = \frac{in_i}{\sum_{i=1}^{r} in_i}, \quad i = 1,...r, \qquad E(Y) = \frac{\sum_{i=1}^{r} i^2 n_i}{\sum_{i=1}^{r} in_i}.$$
10. $0 \le p_X(i) \le 1, \quad i = 1,\cdots,4$ yields $-\dfrac{1}{3} \le d \le \dfrac{1}{4}.$

    $$E(X) = \frac{10-9d}{4}, E(X^2) = \frac{30-47d}{4}, V(X) = \frac{20-8d-81d^2}{16}.$$

    V(X) is minimized at $d = \dfrac{1}{4}.$
11. $\sum_{x=0}^{\infty} p_X(x) = k$, so k = 1.

    $$
    \begin{aligned}
    F(x) &= 0, && \text{if} \quad x < 0,\\
    &= 1/2, && \text{if} \quad 0 \le x < 1,\\
    &= 2/3, && \text{if} \quad 1 \le x < 2,\\
    &\vdots\\
    &= n/(n+1), && \text{if} \quad n-1 \le x < n,\\
    &\vdots
    \end{aligned}
    $$

    E(X) does not exist. Any M between 0 and 1 is a median.
12. Let X denote the scores on IQ test.
    P(X < 52 or X > 148) = P(|X – 100| > 48)
    $\le \dfrac{V(X)}{(48)^2} = \dfrac{1}{9}.$