MA 20104 Probability and Statistics
Assignment No. 3

1. Ruby and Mini tied for the first place in a beauty contest. The winner is to be decided by the majority opinion of a panel of three judges chosen at random from a group of seven judges. If four of these judges favour Ruby and three favour Mini, what is the probability that Ruby will be declared the winner.

2. In a precision bombing attack there is a **50%** chance that a bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give at least **99%** chance of completely destroying the target?

3. It is known that any electronic device produced by a certain company will be defective with probability 0.1, independently of any other item. What is the probability that in a sample of three items, at most one will be defective?

4. Let X follow a Binomial distribution with mean 8 and standard deviation 2. Find $P(X \geq 3)$.

5. A communication system consists of n components each of which function independently with probability p. The entire system will be able to operate effectively, if at least one-half of its components function. For what values of p, a 5-component system more likely to operate effectively than a 3-component system?

6. Let the mgf of a random variable X be given by

$$M_X(t) = \frac{2\,e^t}{7 - 5\,e^t}, \quad t < \log_e\left(\frac{7}{5}\right)$$

   Find $P(X > 7 \mid X > 5)$.

7. The diskettes produced by a certain company are defective with probability 0.01, independently of each other. The company sells the diskettes in packs of size 10 and offers a money-back guarantee if more than one of the 10 diskettes in the pack is found to be defective. If you buy 3 packs, what is the probability that at most one pack will be returned?

8. The probability that an experiment has a successful outcome is 0.8. The experiment is to be repeated until three successful outcomes have occurred. Let X be the number of repetitions required in order to have 3 successful outcomes? What is the probability that at least 5 repetitions will be required?

9. An electronic store has twenty TV sets. Five of these have some manufacturing defect. A customer randomly selects four TV sets. Find the probability that the sample will have exactly one defective.

10. The number X of computers that a hardware store sells in a week obeys the Poisson distribution with $\lambda = 2$. The profit on each computer is Rs. 2000.00. If at the time of opening the store 10 computers are in stock (with no replenishment during the daytime), the profit from the sale of the computers during the day is Y = 2000 min(X, 10). Find the probability distribution of Y.

11. The number of times that an individual contracts cold in a given year is a Poisson random variable with parameter $\lambda = 3$. Suppose that a new drug has been just marketed that reduces the parameter λ to 2 for 75% of the population. For the other 25% of the population the drug has no appreciable effect on the cold. If an individual tries the drug for a year and has no cold in that time, how likely is it that the drug is beneficial for him?

12. A large microprocessor chip contains multiple copies of circuits. If a circuit fails, the chip knows it and knows how to select the proper logic to repair itself. The average number of defects per chip is 300. What is the probability that no more than 4 defects will be found in a randomly selected area that comprises 2% of the total surface area?

13. A boy and a girl decide to meet between 5 and 6 p.m. in a park. They decide not to wait for the other for more than 20 minutes. Assuming arrivals to be independent and uniformly distributed, find the probability that they will meet.

14. Find $Var(X)$, if the mgf of the random variable X is

$$M_X(t) = \begin{cases} \dfrac{e^t(e^{10t}-1)}{10\,(e^t-1)}, & \text{if } t \neq 0 \\ 1, & \text{if } t = 0 \end{cases}.$$

15. A contractor has found through experience that the low bid for a job is a uniform random variable on ($\frac{3}{4}$C, 2C), where C is the contractor's cost estimate (no profit, no loss) of the job. The profit is defined as zero if the contractor does not get the job and as the difference between his bid and his cost estimate C if he gets the job. What should he bid (in terms of C) in order to maximize his expected profit?

16. A small industrial unit has **10** bulbs whose lifetimes are independent exponentially distributed with mean **50** hours. If all the bulbs are used at a time, find the probability that even after **100** hours there are at least two bulbs working.

17. The time to failure in months, X, of the light bulbs produced at two manufacturing plants A and B obeys exponential distribution with means 5 and 2 months respectively. Plant B produces three times as many bulbs as plant A. The bulbs indistinguishable to eye are intermingled and sold. What is the probability that a bulb purchased at random will burn at least 5 months?

18. The motherboard of a new CPU is guaranteed for 6 months. The mean life of a motherboard is estimated to be two years, and the time to failure has an exponential density. The realized profit on a new CPU is Rs. 5,000.00. Including costs of parts and labour the dealer must pay Rs. 2000.00 to repair each failure. Assuming at most one failure in the first 6 months, find the expected profit (in Rs.) per CPU.

19. A series system has n independent components. For i = 1,…, n, the lifetime X_i of the i^{th} component is exponentially distributed with parameter λ_i. If the system has failed before time t what is the probability the failure was caused only by component j (j = 1,…,n).

20. A small shopping mall has five air-conditioner (AC's). The lifetimes of ACs follow independent and identical exponential distributions with mean 100 hours. If all AC's are used simultaneously, find the probability that after 100 hours there are at least two AC's in working condition.

21. The time (in minutes) between arrivals of customers at an ATM machine is exponentially distributed random variable with mean 10 minutes. What is the probability that starting at 9:00 a.m., the third customer will arrive within fifteen minutes?

22. The lead time for orders of diodes from a certain manufacturer is known to have a gamma distribution with a mean of 20 days and a standard deviation of 10 days. Determine the probability of receiving an order within 15 days of placement date.

23. The life (in years) of an electronic equipment is known to follow a gamma distribution. The equipments produced by manufacturer 'A' have mean 4 and variance 8 whereas those produced by manufacturer 'B' have mean 2 and variance 4. An organization procures 75% units of the equipment from 'A' and 25% from 'B'. A unit selected at random is found to be working after 12 years. Find the probability that it was produced by 'A'.

24. A computer lab has three printers. Printer I handles 30% of all jobs and its printing time follows an exponential distribution with mean 3 minutes. Printer II also handles 30% of all jobs and its printing time follows a gamma distribution with mean time 2 minutes and variance 2. Printer III handles remaining 40% of all the jobs and its printing time follows a uniform distribution between 0 and 4 minutes.

Find the probability that a randomly selected job will be printed in less than one minute.

25. The lifetime X in hours of a component is modelled as a Weibull distribution with $\beta = 2$. Starting with a large number of components it is observed that 15% of the components that have lasted 90 hours fail before 100 hours. Determine the parameter α. Further determine the probability that a component is working after 80 hours.

26. Under a certain complicated birth situation, the mortality rate of a new born child is given by $Z(t) = 0.5+2t$, $t > 0$. If the baby survives to age one, find the probability that he/she will survive to age 2.

27. In an examination, a student is considered to have failed, secured second class, first class and distinction, according as he scores less than 45%, between 45% and 60%, between 60% and 75% and above 75% respectively. In a particular year 10% of students failed in the examination and 5% students got distinction. Assuming the marks to be normally distributed, find the percentage of students who get first class and second class respectively.

28. In an industrial process the diameter of a ball bearing is an important component. The buyer sets specifications on the diameter to be 3.0 ± 0.01 cm. The diameter has a normal distribution with mean 3 cm. and s.d. 0.005 cm. On the average how many manufactured balls will be scrapped?

29. The width of a duralumin forging is (in inches) normally distributed with $\mu = 0.9$ and $\sigma = 0.003$. The specification limits were given as 0.9 ± 0.005. What percentage of forgings will be defective? What is the maximum allowable value of σ that will permit no more than 1 defective in 100 when the widths are $N(0.9, \sigma^2)$?

30. The height a university high jumper will clear, each time he jumps, is a normal r.v. with mean 2 meters and s.d. 10 cm. What is the greatest height that he will jump with probability 0.95? What is the height that he will clear only 10% of the time?

31. If a set of marks on a Statistics exam is approximately N(74, 62.41), find

    a) the lowest passing grade if the lowest 10% of the students are given F’s;

    b) the highest B if the top 5% of the students are given A’s;

    c) the lowest B if the top 10% of the students are given A’s and the next 25% are given B’s.

32. The diameters X of a ball-bearing are distributed normally with mean μ and standard deviation unity, If X lies in the specification limits of 6 to 8 inches, a profit of Rs. C_0 is gained. However, in case X < 6 or X > 8, there is a loss of Rs. C_1 or Rs. C_2 respectively. Find the value of μ that maximizes the expected profit.

33. IQ levels of the candidates for a particular job selection are normally distributed with mean 90 and standard deviation 5. Find the approximate probability that a randomly selected candidate has IQ level between 85 and 95. Suppose four candidates are randomly selected. Find the probability that at least two of them have IQ levels between 85 and 95.

34. A production company has 200 machines and they operate independently. The time to failure for each machine is given by a random variable X , measured in years with pdf given as

$$f(x) = \begin{cases} \frac{2}{x^3}, & \text{if } x > 1 \\ 0, & \text{if } x \le 1 \end{cases}$$

Use binomial approximation to normal to find the probability that at least 60 machines will be working for more than two years. (You have to use tables of normal cdf).

35. In a probability and statistics class, the total number of students is 200. A professor gives a question to the students as a surprise test. The probability that a randomly selected student can solve the question is 0.5. What is the probability that at least 110 students in the class cannot solve the Question? Use normal approximation to binomial (without continuity corrections).

36. Trains arrive and depart at Kanpur railway station according to a Poisson process at a rate one per three minutes. What is the probability that between 2:00 p.m. to 3:00 p.m. the number of trains arriving or departing is at least 17 and not more than 25? Use normal approximation with continuity corrections.

37. Let Y denote the diameter in mm. of certain type of nuts. Assume that Y has a log-normal distribution with parameters μ = 0.8 and σ = 0.1. Find the probability that a randomly selected nut has diameter more than 2.7 mm. Between what two values will Y fall with probability 0.95?

38. A random variable X has a beta distribution with mean $\frac{2}{3}$ and the variance is $\frac{1}{18}$. Find $P(0.2 < X < 0.5)$.

39. Let X follow a zero truncated Poisson distribution with the probability mass function given by

$$P(X = x) = \frac{1}{1-e^{-\lambda}} \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 1, 2, 3, \ldots$$

Find $E\left(\frac{1}{1+X}\right)$.

40. Let the moment generating function of a random variable X be given by

$$M_X(t) = \frac{e^t}{3-2e^t}, \quad t < \log_e 3 - \log_e 2.$$

Find $E(X), V(X)$ and $P(3 < X \leq 5)$.

41. Five balls are drawn without replacement from an urn containing 3 white and 7 red balls. Write in detail the probability distribution of the number of the white balls among these. Hence derive the mean and median of this distribution.

42. For a certain job, Roshni is sending many applications. She estimates that the probability of one application being successfully accepted is 0.3. How many applications she needs to send so that probability of at least one acceptance is more than 0.95?

43. In independent Bernoullian trials (with success probability $p, 0 < p < 1$), let P_n denote the probability of even number of successes (0 is assumed to be even) in n trials. Prove that

$$P_n = p\left(1 - P_{n-1}\right) + \left(1 - p\right) P_{n-1}, \; n \geq 1.$$

Using induction show that

$$P_n = \frac{1+(1-2p)^n}{2}.$$

44. Four percent pens produced by a company are defective. Pens are packed in boxes of each containing 150 pens. The company gives the guarantee that at most 2 pens in a box will be defective. Using the Poisson approximation to the binomial, find the probability that a box will fail to meet the guarantee.

45. The number of surface defects in metal sheets produced at Plant P_1 has a Poisson distribution with average 0.75 flaws per square meter of metal sheet. Similarly the number of surface defects in metal sheets produced at Plant P_2 has a Poisson distribution with average 1.75 flaws per square meter of metal sheet. A car manufacturing company procures 60% of its metal sheet from plant P_1 and 40% from Plant P_2. A randomly selected metal sheet of 2 sq mt is found to have one flaw. Find the probability that it was produced by plant P_1.