Proability and Statistics

MA20205

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Lecture 9

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Special continuous distributions

Exponential random variable The pdf of the exponential random variable X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} \text{ if } x \geq 0 \\ 0, \text{ otherwise} \end{cases}$$

where $\lambda > 0$, a parameter.

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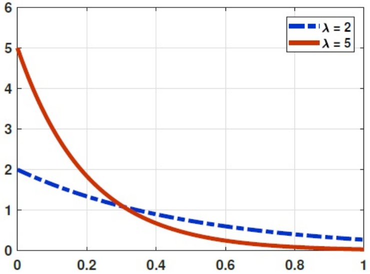

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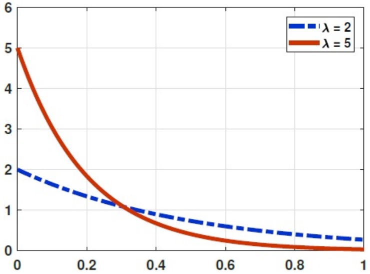

Note Exponential distribution is a special case of Gamma distribution, when $\alpha = 1$

Special continuous distributions

Connection to Poisson distribution From Poisson distribution, note that

$$P(X = 0) = e^{-\lambda}$$

which means that the probability of the amount of time until the event occurs which further means during the waiting period, not a single event has happened is $e^{-\lambda}$.

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Define a random variable T for the waiting time until the first event Then

$$\begin{aligned}
& P(\text{Nothing happens during } t \text{ time units}) \\
= \; & P(X = 0 \text{ in the first time unit}) \cdot P(X = 0 \text{ in the second time unit}) \cdot \ldots \\
& \ldots \cdot P(X = 0 \text{ in the } t\text{-th time unit}) \\
= \; & e^{-\lambda t} = P(T > t)
\end{aligned}$$

where X follows Poisson distribution.

Special continuous distributions

Then

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Memoryless property of exponential distribution: Note that

$$P(T > m+n | T > m) = \frac{P(T > m+n)}{P(T > m)} = \frac{e^{-\lambda(m+n)}}{e^{-\lambda m}} = e^{-\lambda n} = P(T > n).$$

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Let us introduce another random variable T for wait time until the k-th event, where arrival of events is modeled by the Poisson process with rate λ.

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Therefore

$$
\begin{aligned}
F(t) = P(T \leq t) &= 1 - P(T > t) \\
&= 1 - P(0, 1, 2, k-1 \, \text{events in} \, [0,\ t]) \\
&= 1 - \sum_{x=0}^{k-1} \frac{(\lambda t)^x e^{-\lambda t}}{x!}
\end{aligned}
$$

Special continuous distributions

From the CDF $F(t)$, the pdf of T is given by

$$\begin{aligned} f(t) &= \frac{d}{dt}F(t) \\ &= \lambda e^{-\lambda t} - \frac{d}{dt}\left(\sum_{x=0}^{k-1} \frac{(\lambda t)^x e^{-\lambda t}}{x!}\right) \end{aligned}$$

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Performing term by term derivatives, we obtian

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f(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{k-1}}{(k-1)!}
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$$f(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{k-1}}{(k-1)!}$$

Conclusion: If arrivals of events follow the Poisson process with rate λ, the wait time until $k = \alpha$ arrivals follows Gamma$(k = \alpha, \lambda = \beta)$.

Special continuous distributions

Chi-square random variable A random variable X is said to have chi-square distribution with r degrees of freedom if the pdf is given by

$$f(x) = \begin{cases} \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}} \text{ if } 0 < x < \infty \\ 0, \text{ otherwise} \end{cases}$$

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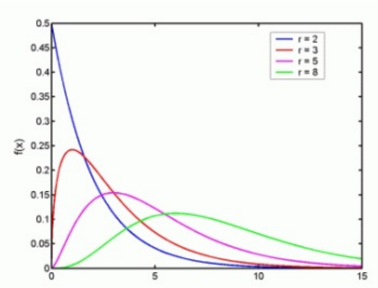

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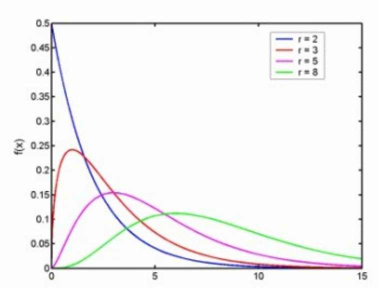

Note chi-square distribution is a special case of Gamma distribution, when $\alpha = \frac{\alpha}{2}, \beta = \frac{1}{2}$. If $r \to \infty$ then chi-square tends to normal distribution

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Weibull random variable A random variable X is said to have Weibull distribution if the pdf is given by

$$f(x) = \begin{cases} \alpha\beta x^{\alpha-1} e^{-\beta x^{\alpha}} \text{ if } 0 < x < \infty \\ 0, \text{ otherwise} \end{cases}$$

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Homework Draw the pdf of Weibull and decide which one is the shape parameter and which one is the scale parameter

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Homework Draw the pdf of Weibull and decide which one is the shape parameter and which one is the scale parameter
Note Setting $\alpha = 1$ it reduces to the exponential distribution

Special continuous distributions

Properties of Weibull(α, β)

(a)

$$E(X) = \left(\frac{1}{\beta}\right)^{\frac{1}{\alpha}} \left(1 + \frac{1}{\alpha}\right)$$

(b)

$$\mathrm{Var}(X) = \left(\frac{1}{\beta}\right)^{\frac{2}{\alpha}} \left\{\Gamma\left(1 + \frac{2}{\alpha}\right) - \left(1 + \frac{1}{\alpha}\right)^2\right\}$$

Special continuous distributions

Recall: Beta function Let $\alpha > 0$ and $\beta > 0$. Then

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

is called the beta function.

Special continuous distributions

Recall: Beta function Let $\alpha > 0$ and $\beta > 0$. Then

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Beta in terms of gamma:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

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Properties of beta function

(a) $B(\alpha, \beta) = B(\beta, \alpha)$

(b) if α, β are positive integers then

$$B(\alpha, \beta) = \frac{(\alpha-1)(\beta-1)}{(\alpha+\beta-1)(\alpha+\beta-2)} B(\alpha-1, \beta-1)$$

Special continuous distributions

Beta distribution: A random variable is said to have beta distribution if the pdf is of the form

$$f(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \text{ if } 0 < x < 1 \\ 0, \text{ otherwise} \end{cases}$$

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Special continuous distributions

Properties of beta distribution:

(a) Shapes

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(b) $E(X) = \frac{\alpha}{\alpha+\beta}$

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(b) $E(X) = \frac{\alpha}{\alpha+\beta}$

(c) $\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Special continuous distributions

Properties of beta distribution:

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(b) $E(X) = \frac{\alpha}{\alpha+\beta}$

(c) $\text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Note Beta reduces to Uniform distribution over (0, 1) if $\alpha = \beta = 1$

Special continuous distributions

Gaussian random variable (normal random variable): A random variable X is said to have Gaussian distribution if the pdf is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, -\infty < x < \infty$$

where $\mu, \sigma \in \mathbb{R}$.

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Observation

(a) A Gaussian random variable is symmetric about μ

(b) If $\mu = 0$, then $f(x)$ is an even function.

(c) If σ is very small, it is possible to have $f(x) > 1$, however integration over $\mathbb{R}$ is 1

Special continuous distributions

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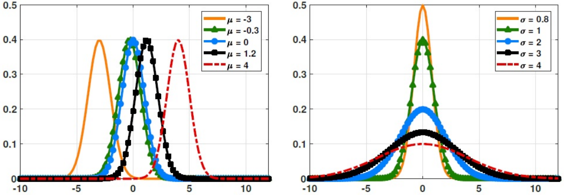

μ changes, $\sigma = 1$ $\qquad$ $\mu = 0$, σ changes

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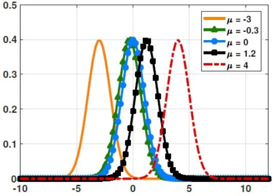

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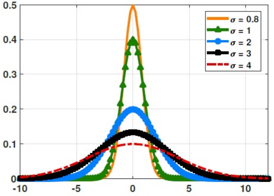

$\mu = 0$, σ changes

Properties of $\mathcal{N}(\mu, \sigma^2)$

(a) $E(X) = \mu$

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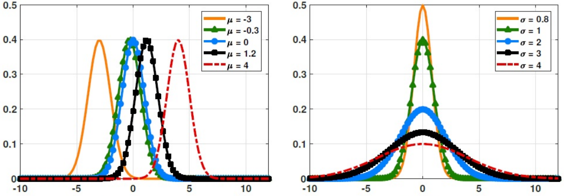

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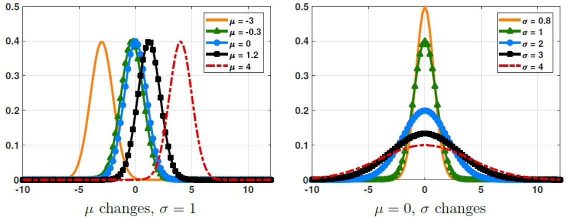

Properties of $\mathcal{N}(\mu, \sigma^2)$

(a) $E(X) = \mu$

(b) $\mathrm{Var}(X) = \sigma^2$

(c) $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

Special continuous distribution

Standard normal distribution A normal random variable X is said to have standard Gaussian/normal distribution is given by

$$\Phi(x) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty$$

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Theorems Let $X = \mathcal{N}(\mu, \sigma^2)$.

(a) Then $Z = \dfrac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$

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Theorems Let $X = \mathcal{N}(\mu, \sigma^2)$.

(a) Then $Z = \dfrac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$

(b) $\left(\dfrac{X - \mu}{\sigma}\right)^2 \sim \chi^2(1)$

Special continuous distribution

$$
\begin{aligned}
F(z) &= P(Z \leq z) \\
&= P\left(\frac{X-\mu}{\sigma} \leq z\right) \\
&= P(X \leq \sigma z + \mu) \\
&= \int_{-\infty}^{\sigma z+\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx \\
&= \int_{-\infty}^{z} \frac{1}{\sigma\sqrt{2\pi}} \sigma e^{-\frac{1}{2}w^2},\ w = \frac{x-\mu}{\sigma}.
\end{aligned}
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&= \int_{-\infty}^{z} \frac{1}{\sigma\sqrt{2\pi}} \sigma e^{-\frac{1}{2}w^2}, \; w = \frac{x-\mu}{\sigma}.
\end{aligned}
$$

$f(z) = F'(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$