Proability and Statistics

MA20205

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Lecture 8

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Special discrete distributions

From Poisson process to Poisson distribution

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Poisson process (PP)

A Poisson Process is a model for a series of discrete events where the average time between events is known, but the exact timing of events is random. The arrival of an event is independent of the event before (waiting time between events is memoryless).

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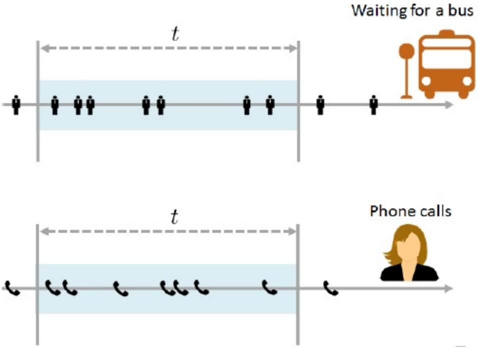

Special discrete distributions

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2. The average rate (events per time period) is constant.
3. Two events cannot occur at the same time.

Note Events are not simultaneous — means we can think of each sub-interval of a Poisson process as a Bernoulli Trial, that is, either a success or a failure. (Bus arrivals need not perfectly follow all the criteria)

Special discrete distributions

Poisson Distribution probability mass function gives the probability of observing k events in a time period given the length of the period and the average events per time:

$$P(k \text{ events in time period}) = e^{-\frac{\text{events}}{time}\times\text{time period}} \frac{\left(\frac{\text{events}}{time} \times \text{time period}\right)^{k}}{k!}$$

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Setting

$$\begin{aligned} \lambda &= \frac{\text{events}}{time} \times \text{time period} \\ &= \text{expected number of events in the time interval (time period)} \\ &= \text{rate parameter} \end{aligned}$$

we obtain the desired pmf.

Click the link: The Waiting Time Paradox, or, Why Is My Bus Always Late?

Special discrete distributions

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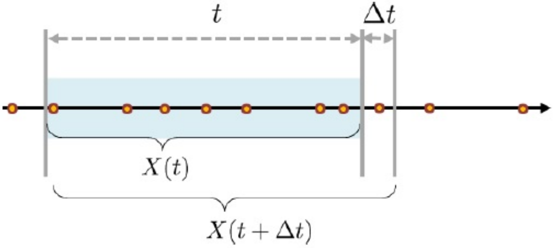

For small $\triangle t$,

$$P\left(X(t + \triangle t) - X(t) = 1\right) = \lambda \cdot \triangle t$$

where λ is the rate of occurance of events. Here $X(t + \triangle) - X(t)$ can be thought as a Bernoulli trial.

Special discrete distributions

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Setting $\triangle t\to 0$ we obtain the ODE

$$\frac{d}{dt}P[X(t)=k]=\lambda\left(P[X(t)=k-1]-P[X(t)=k]\right)$$

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Claim: $P(X(t)=k)=\frac{(\lambda t)^k}{k!}e^{-\lambda t}$ is the solution of the ODE.

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$$\begin{aligned}\frac{d}{dt}\left(\frac{(\lambda t)^k}{k!}e^{-\lambda t}\right) &= \lambda\frac{(\lambda t)^{k-1}}{(k-1)!}e^{-\lambda t}-\lambda\frac{(\lambda t)^k}{k!}e^{-\lambda t}\\ &= \lambda(P[X(t)=k-1]-P[X(t)=k])\end{aligned}$$

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$$= \lambda(P[X(t) = k-1] - P[X(t) = k])$$

Setting $t = 1$ we obtain the Poisson distribution

Special discrete distributions

Binomial approximation to Poisson If n is large in $\text{Bin}(n, p)$ then calculation the probabilities is a prohibited amount of work. For instance, let $n = 3000$, $p = 0.005$ and try to calculate $P(X = 29)$ when $X \sim \text{Bin}(n, p)$.

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Setting $n \to \infty$ and $p \to 0$ with $\lambda = np$, a constant, the binomial probability is given by

$$\begin{aligned}\binom{n}{k} p^k (1-p)^{n-k} &= \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \\ &\to \frac{\lambda^k}{k!} e^{-\lambda} \text{ as } n \to \infty\end{aligned}$$

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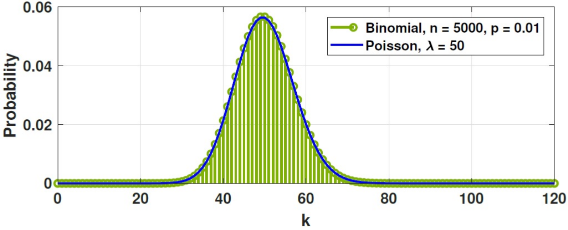
Question How does this derivation relate to Poisson Process?

Special discrete distributions

Comparing Poisson with Binomial



Special continuous distributions

Continuous uniform random variable The pdf of the uniform random variable X is

$$f(x) = \begin{cases} \dfrac{1}{b-a} \text{ if } x \in [a, \ b] \\ 0, \text{ otherwise} \end{cases}$$

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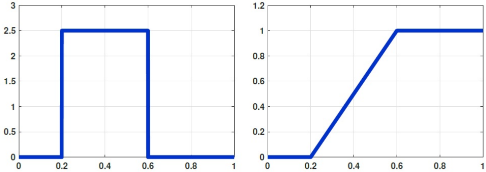
$\text{Unif}(0.2, 0.6)$ looks like



(a) PDF (b) CDF

Special continuous distributions

Properties of Unif(a, b) :

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(c)

$$M(t) = \begin{cases} 1, \text{ if } t = 0 \\ \dfrac{e^{tb} - e^{ta}}{t(b-a)} \text{ if } t \neq 0 \end{cases}$$

Special continuous distributions

Recall: Gamma function

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx,$$

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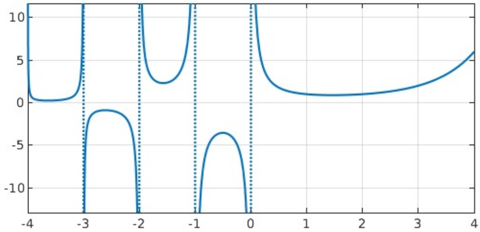

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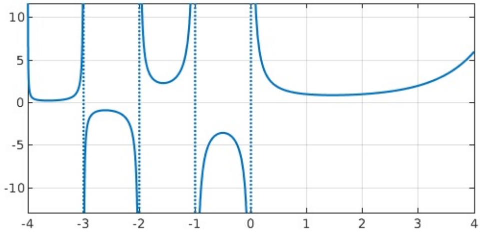

Note There is an alternative way to define Gamma function when $z \leq 0$.
The above expression on the rhs is also known as Euler's second integral

Special continuous distributions

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(e) $\Gamma(n+1) = n!$ for any positive integer n

Special continuous distributions

Gamma distribution A random variable X has Gamma distribution if the pdf is given by

$$f(x) = \begin{cases} \dfrac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \text{ if } 0 < x < \infty \\ 0, \text{ otherwise} \end{cases}$$

where $\alpha > 0, \beta > 0$

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where $\alpha > 0, \beta > 0$
Then we write $X \sim \text{Gamma}(\alpha, \beta)$

Special continuous distributions

Shape parameter: α

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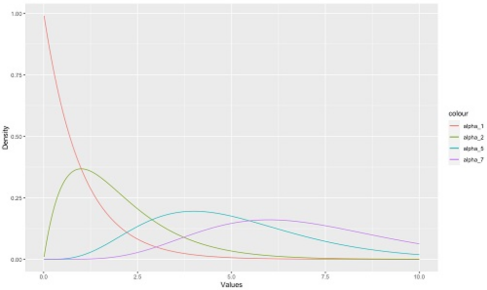
The pdf for Gamma$(\alpha, 1)$ looks like



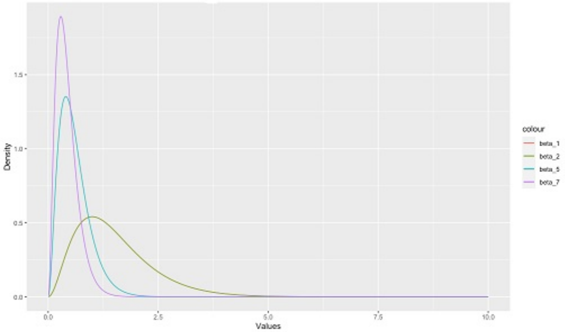
Special continuous distributions

Scale parameter: β

Special continuous distributions

Scale parameter: β

The pdf for Gamma$(3, \beta)$ looks like



Special continuous distributions

pdf of Gamma(α, β)

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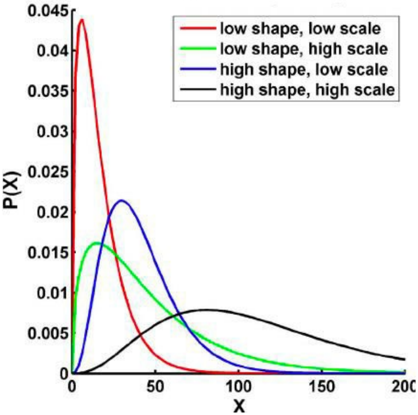

Special continuous distributions

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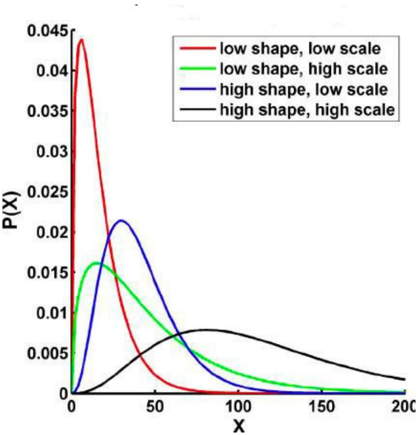

Properties:

$$E(X) = \frac{\alpha}{\beta}, \ \text{Var}(X) = \frac{\alpha}{\beta^2}$$