Proability and Statistics

MA20205

Bibhas Adhikari

Autumn 2022-23, IIT Kharagpur

Lecture 7

September 5, 2022

Special distributions

Review of last week

- pdf and cdf for continuous random variables

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  - percentile, median, mode

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- moment generating function
- special discrete distributions
  - Bernoulli and Binomial

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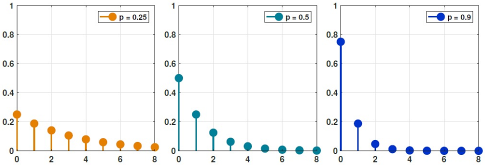
The pmf of Geometric(p) looks like:



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Observation Geometric(p) explains the phenomena of a binary experiment (Bernoulli trials) until the success is reached (the trial number on which the first success occurs)

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$$P(X > m + n | X > n) = P(X > m)$$

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Although the Bin(n, p) and Geo(p) are based on sequence of Bernoulli trials, the fundamental difference is: the number of trials in Bin(n, p) is predetermined whereas it is the random variable in the case of Geo(p)

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Negative Binomial Distribution A random variable follows negative binomial if the pmf is

$$f(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, k = r, r+1, \ldots$$

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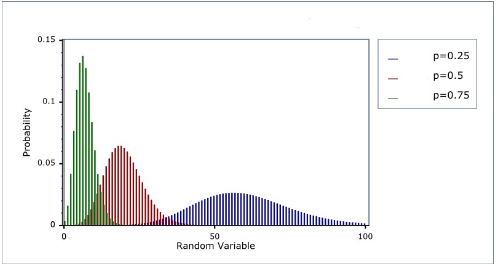
The pmf of NBIN$(20, p)$ looks like:



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NBIN is called Pascal distributions or binomial waiting-time distributions

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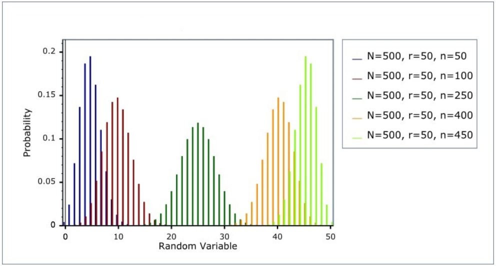

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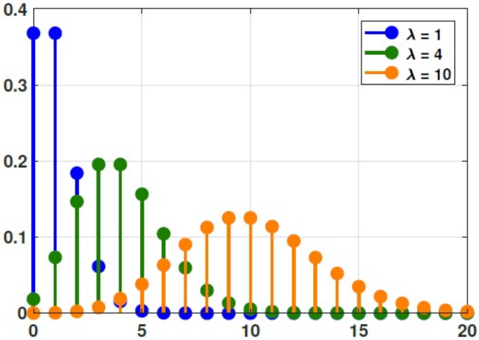
The pmf of Poisson(λ) looks like:



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(d) $M(t) = e^{\lambda(e^t - 1)}$

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