Proability and Statistics

MA20205

Bibhas Adhikari

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Lecture 6
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Key parameters for analyzing random variables

Observation

(1) $\frac{d}{dt}M(t) = \frac{d}{dt}E(e^{tx}) = E\left(\frac{d}{dt}e^{tX}\right) = E\left(Xe^{tX}\right)$

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Problems...

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Proof follows from applying the above

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Skewness: measures the asymmetry of the distribution

$$\gamma = E\left(\left(\frac{X-\mu}{\sigma}\right)^3\right)$$

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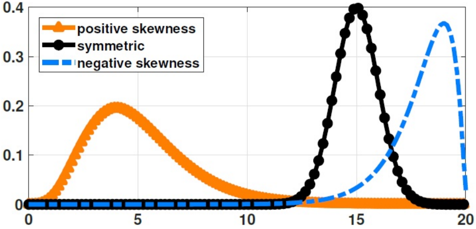

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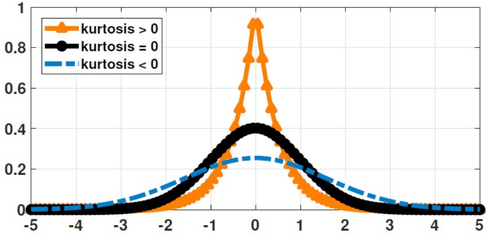

Positive and negative kurtosis is defined based on the kurtosis of Gaussian random variable

Special discrete distributions

Bernoulli random variable Let X be a random variable with range space $R_X = \{0, 1\}$. Then we say X to have Bernoulli distribution if the pmf of X is

$$f(0) = 1 - p \text{ and } f(1) = p$$

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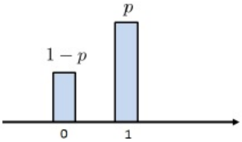

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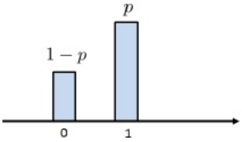

Then we denote $X \sim \text{Bernoulli}(p)$

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Binomial random variable A random variable X is said to have binomial distribution if the pmf of X is

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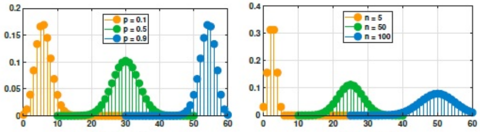
The pmf of Binomial(n, p) looks like:



(a) $n = 60$ (b) $p = 0.5$

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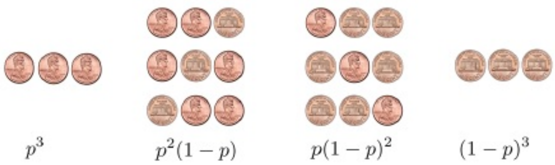

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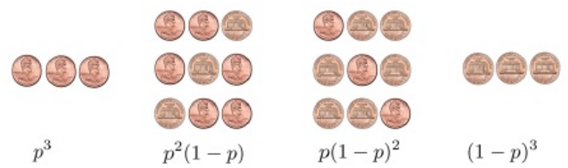

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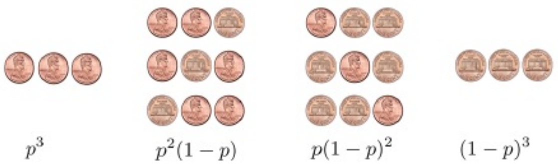

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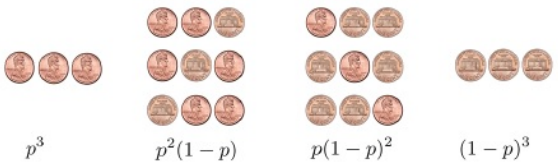

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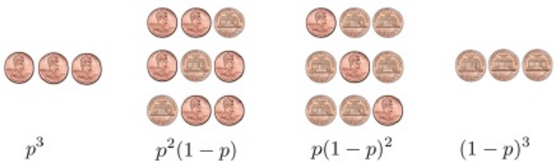

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