Proability and Statistics

MA20205

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Autumn 2022-23, IIT Kharagpur

Lecture 5

August 29, 2022

Discrete random variables

Review of last week

- Random variables: $X : S \to \mathbb{R}$

Discrete random variables

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  - cumulative density/distribution function (cdf):

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Discrete random variables

Cumulative distribution function

Let X be a discrete random variable with range space $R_X = \{x_1, x_2, \ldots\}$. Then

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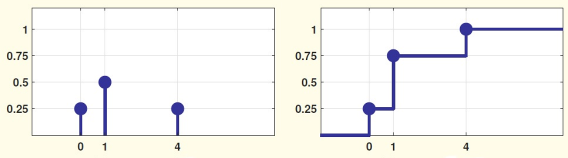

(a) PMF (b) CDF

Discrete random variables

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Properties of cdf

(a) $F(-\infty) = 0$

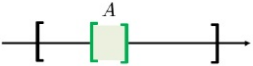
(b) $F(\infty) = 1$

(c) $F(x)$ is an increasing function: if $x < y$ then $F(x) < F(y)$

Continuous random variables

examples:

(1) Suppose we are concerned with the possibility that an accident will occur on a highway that is 1000 kilometers long and that we are interested in the probability that it will occur at a given location, or perhaps on a given stretch of the road. Then the sample space is the (continuous) interval [0, 1000]



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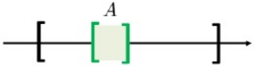
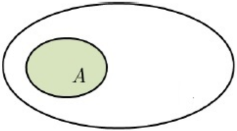


(2) Throwing a dart



Continuous random variables

Properties of probability density function (pdf)

If a function $f : \mathbb{R} \to \mathbb{R}$ represents a pdf of a continuous random variable X then

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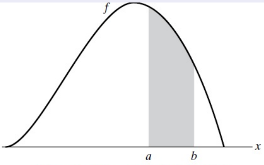
pdf

If f is the pdf of a r.v. X then for an event $A = [a, b]$:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$



$P(a \leq X \leq b)$ = area of shaded region

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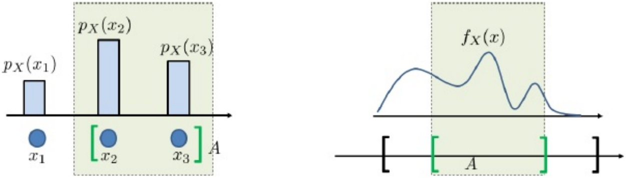
pmf and pdf



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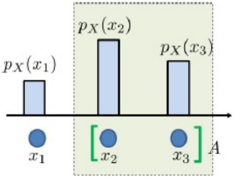

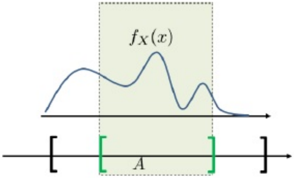

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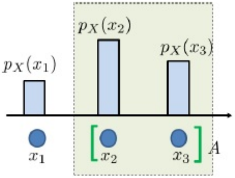

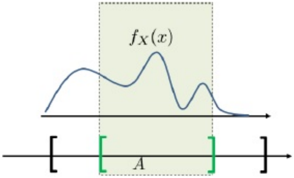

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$$f(x) = \frac{1}{2\pi}, 0 \leq x \leq 2\pi, \; P(0 \leq X \leq \pi/2) = \frac{1}{4}$$

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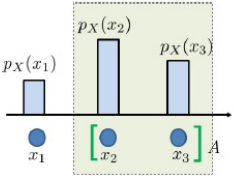

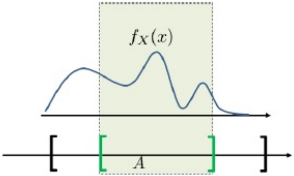

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cdf

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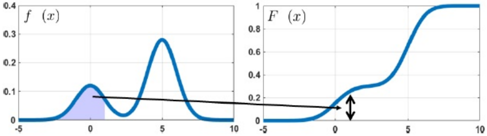

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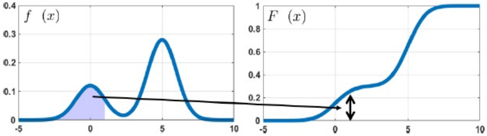

properties

(1) non-decreasing

(2) $F(\infty) = 1$ and $F(-\infty) = 0$

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Proposition:

(1) $P(X < x) = F(x)$

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Question How to obtain pdf from cdf?

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The proof follows from fundamental theorem of calculus:

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(\int_{-\infty}^{x} f(t)dt\right) = f(x)\frac{dx}{dx} = f(x).$$

Key parameters for analyzing continuous random variables

Percentile

Let p be a number between 0 to 1. Then $100p$-th percentile of a continuous r,v, is a real number q such that

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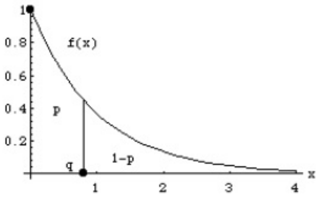

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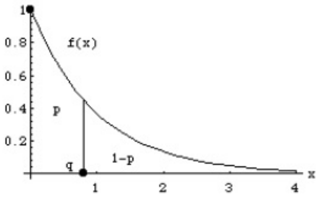

Problems...

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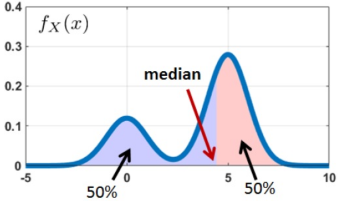

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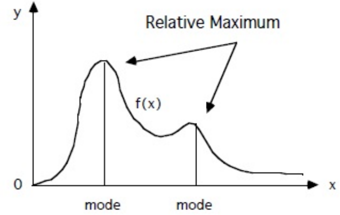

Key parameters for analyzing random variables

Moments

The *n*th moment about the origin of a random variable X is defined by

$$E(X^n) = \begin{cases} \sum_{x \in R_X} x^n f(x) \text{ if } X \text{ is discrete} \\ \\ \int_{-\infty}^{\infty} x^n f(x) dx \text{ if } X \text{ is continuous} \end{cases}$$

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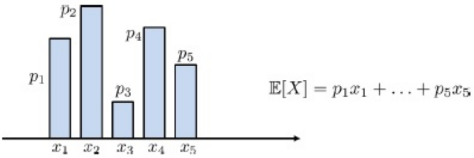

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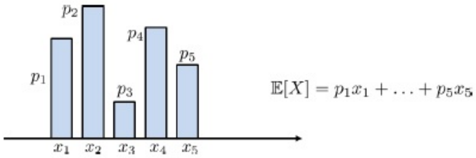

Question Why are they called moments?

Key parameters for analyzing random variables

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Existence of expectation: Consider a pmf

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$$f(k) = \frac{6}{\pi^2 k^2}, k = 1, 2, \ldots$$

Note that $\sum_k \frac{1}{k^2} = \frac{\pi^2}{6}$. Then

$$\begin{aligned} E(X) &= \sum_k k \frac{6}{\pi^2 k^2} \\ &= \frac{6}{\pi^2} \sum_k \frac{1}{k} \to \infty \end{aligned}$$

Thus $E(X)$ does not exist.

Key parameters for analyzing random variables

Properties of expectation: Let X be a randomvariable with pmf/pdf $f(x)$. Then

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(3) Scaling: $E(cX) = cE(X)$

Key parameters for analyzing random variables

Spread of the distribution

The variance of a random variable X is defined as

$$\sigma^2 = \text{Var}(X) = E\left[(X-\mu)^2\right] = E(X^2) - \mu^2.$$

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(2) The spread can also be measured by the area between two values

(3) Conclusion: σ controls the area between two values

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Markov inequality: Let X be a non-negative random variable. Then

$$P(X \geq \epsilon) \leq \frac{E(X)}{\epsilon}$$

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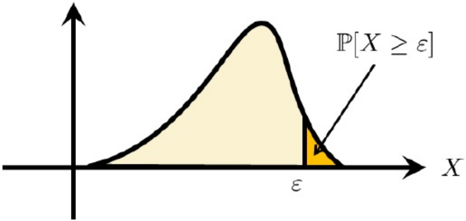

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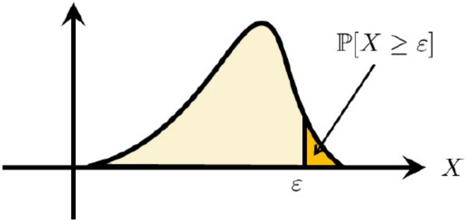

The proof:

$$\epsilon P(X \geq \epsilon) = \int_{\epsilon}^{\infty} \epsilon f(x)dx \leq \int_{\epsilon}^{\infty} xf(x)dx \leq \int_{0}^{\infty} xf(x)dx = E(X)$$

Key parameters for analyzing random variables

Chebyshev's inequality: Let X be a random variable with mean μ. Then for any $\epsilon > 0$,

$$P(|X - \mu| \geq \epsilon) \leq \frac{\mathrm{Var}(X)}{\epsilon^2}$$

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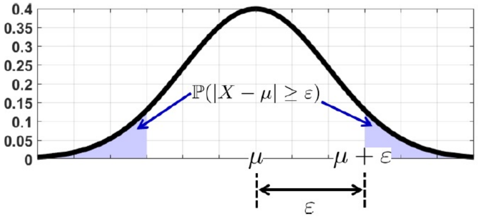

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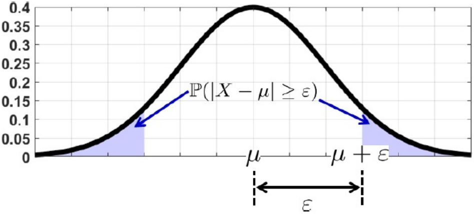

Proof: $P(|X - \mu| \geq \epsilon) = P(|X - \mu|^2 \geq \epsilon^2) \leq \frac{E((X - \mu)^2)}{\epsilon^2} = \frac{\text{Var}(X)}{\epsilon^2}$

Key parameters for analyzing random variables

Alternative: Setting $\epsilon = k\sigma$,

$$P(|X - \mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2},$$

for any positive constant k

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$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Key parameters for analyzing random variables

Moment generating function (mgf): generating moments of a random variable

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mgf

Let X be a random variable with pdf $f(x)$. Then the function $M : \mathbb{R} \to \mathbb{R}$ given by

$$M(t) = E\left(e^{tX}\right)$$

is called mgf of X if the expectation exists in some interval $[-h, h], h > 0$.

Thus

$$M(t) = \begin{cases} \sum_{x \in R_X} e^{tx} f(x) \text{ if } X \text{ is discrete} \\ \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx \text{ if } X \text{ is continuous} \end{cases}$$

Key parameters for analyzing random variables

Observation

(1) $\frac{d}{dt}M(t) = \frac{d}{dt}E(e^{tx}) = E\left(\frac{d}{dt}e^{tX}\right) = E\left(Xe^{tX}\right)$

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Problems...