Proability and Statistics

MA20205

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Lecture 3
August 23, 2022

Review of Lecture 1 and 2

- Random experiment

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- Random experiment
- Sample space

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- Events

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The above notions develop the notion of probability space:

$$(S, \mathcal{F}, P)$$

- Properties of probability measure
- Conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided $P(B) > 0$

Probability

Sequential trials: Let A and B are disjoints events for a sample space. Then

Question: What is $P(A \text{ before } B)$?

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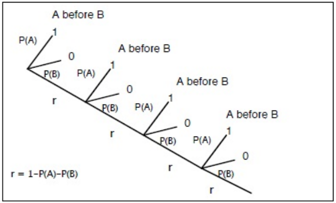

$$\begin{aligned} P(A \text{ before } B) &= P(A) + rP(A) + r^2 P(A) + \dots \\ &= P(A)(1 + r + r^2 + \dots) \end{aligned}$$

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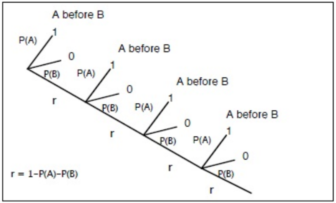

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\begin{aligned}
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&= P(A)\frac{1}{1-r} = \frac{P(A)}{P(A)+P(B)}
\end{aligned}
$$

Probability

Conditional probability interpretation of sequential trial:

$$P(A \,\text{before}\, B) = P(A|A \cup B).$$

Problem.....

Probability

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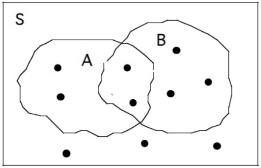
Problem.....

Sampling with/without replacement

The phenomena of an object is selected and then replaced before the next object is selected is called sampling with replacement. Otherwise it is called sampling without replacement.

Problem....

Probability



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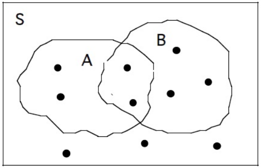


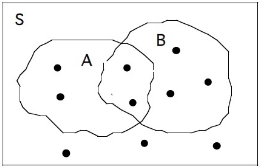
Independent events

Two events A and B are called independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Question: What is the meaning of 'independence' here?

Probability



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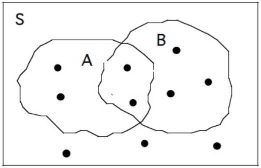

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Observation: If A and B are independent events then

- A^c and B are independent
- A and B^c are independent

Probability

Partition of a set

Let $A_1, A_2. \ldots, A_k$ be a collection of subsets of a set S such that

- $\cup_{i=1}^{k} A_i = S$
- $A_i \cap A_j = \emptyset$ if $i \neq j$.

Then the collection of sets is called a partition of S.

Probability

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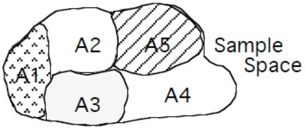

Probability

Theorem: (law of total probability) Let $B_1, B_2, \ldots, B_k$ form a partition of a sample space S with $P(B_i) \neq 0, i = 1, \ldots, k$. Then for any event A of S :

$$P(A) = \sum_{i=1}^{k} P(B_i)P(A|B_i)$$

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The proof follows from the fact (draw the Venn diagram) that

$$A = \cup_{i=1}^{k} A \cap B_i.$$

Probability

Bayes Theorem: Let $B_1, \ldots, B_k$ be a partition of a sample space S with $P(B_i) \neq 0$. Then for any event A of S with $P(A) \neq 0$:

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$$

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and the multiplication rule

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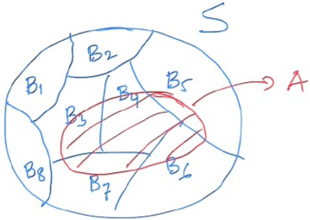
- $P(B_i)$ is called prior probability
- $P(B_i|A)$ is called posterior probability

Problem...



Random Variables

Random variable: mathematizing sample space

A random variable associates sample points to real numbers:

$$X : S \to \mathbb{R}$$

such that for any interval $I \subset \mathbb{R}$, $\{s \in S : X(s) \in I\}$ is an event

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Consider $S = \{H, T\}$, sample space of the coin toss. Are

$$X_1 : \begin{cases} X_1(H) = 1 \\ X_1(T) = 0 \end{cases} \quad X_2 : \begin{cases} X_2(H) = 1 \\ X_2(T) = 2 \end{cases}$$

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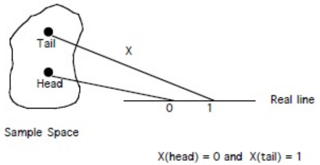
random variables?
How do they differ (representation)?

Random Variables

Things not to forget: A random variable is neither 'random' nor a 'variable'



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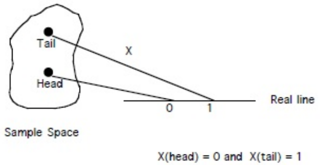

Range space of a random variable

$$R_X = \{X(s) : s \in S\}$$

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Defining events through random variable

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Important notations

$$a \le X \le b \equiv \{s \in S : a \le X(s) \le b\}, a, b \in \mathbb{R}$$

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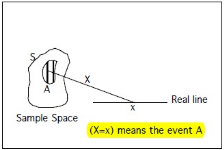

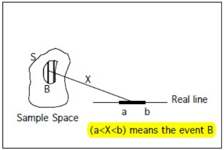

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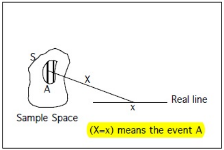

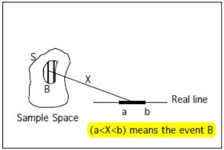

Probability of events of type $X = x$

Example: Suppose an unbiased coin is tossed four times. Let X denote the number of heads in these tosses. Then

$$P(X = 3) \;=$$

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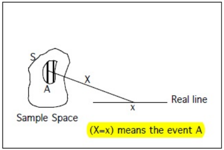

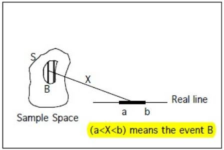

Probability of events of type $X = x$

Example: Suppose an unbiased coin is tossed four times. Let X denote the number of heads in these tosses. Then

$$P(X = 3) = \frac{4}{16}$$